Sunbad

~~24/01/22~~ 24/01/22

Lecture 7

- Next Hw is now up.

Recall Lemma 3.1(b) - Spose ~~if~~ F_*, G_* are free resolutions of M, $\alpha_* : F_* \longrightarrow G_*$ has $\alpha_{-1} = Id_M$. Then α_* induces isomorphisms $\cancel{\alpha^*}$

$$\alpha^* : H^*(G, Q) \longrightarrow H^*(F, Q)$$

Pf (3.1(b)) we showed last time that if $\alpha_*, \beta_* : F_* \longrightarrow G_*$ are extensions of α $[\alpha_{-1} = \beta_{-1} = \alpha]$, then there is a chain homotopy P_* s.t. $\alpha_{k} - \beta_k = P_{k-1} \circ \delta_k + \eta_{k+1} \circ P_k$

[diagram:

$$\begin{tikzcd}
 & F_k \arrow[r, "\delta_k"] \arrow[dl, "P_k"'] \arrow[d, bend right, "\alpha_k"'] \arrow[d, bend left, "\beta_k"] & F_{k-1} \arrow[dl, "P_{k-1}"] \\
G_{k+1} \arrow[r, "\eta_{k+1}"] & G_k &
\end{tikzcd}$$

]

Dualise to find ...

$$\alpha_k^* - \beta_k^* = \varphi_k^* \circ P_{k-1}^* + P_k^* \circ \varphi_{k+1}^*.$$

<u>Thus</u>: P^* is a (co)chain homotopy

from $\alpha^* : C^* \to F^*$

to $\beta^* : C^* \to F^*$

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<u>Thus</u>: α^*, β^* induce the same homomorphisms on cohomology

up to here is a review of the proof of (a).

· <u>Now</u> to apply to the special situation where $\alpha_{-1} = Id_M$.

Let $\beta_* : C_* \to F_*$ be s.t. $\beta_{-1} = Id_M$. Thus

$\alpha_* \circ \beta_* \simeq Id_C$ $\beta^* \circ \alpha^* \simeq Id_{C^*}$

$\beta_* \circ \alpha_* \simeq Id_F$, so: $\alpha^* \circ \beta^* \simeq Id_{F^*}$

induce ~~identities~~ isomorphisms

$$\alpha^*: H^*(G,Q) \longrightarrow H^*(F,Q)$$

$$\beta^*: H^*(F,Q) \longrightarrow H^*(G,Q).$$

□

~~Ass~~

Question Is β not going the other way?

answer ~~If~~ Notation has been abused, β in part (b) is very disserent srom β in part (a).

~~clarisication~~

clarisecation This isomorphism is "natural" by part (a), as is we make disserent choices to build α_*, this gives the same isomorphism α^*.

excercises ① $Ext^k_R(M,Q) \cong 0$ is M is gree.

② $Ext^k_R(A \oplus B, Q) \cong \cancel{Ext^k_R(A \oplus B, Q) \cong Ext^k_R}$

$$Ext^k_R(A,Q) \cong Ext^k_R(B,Q)$$

③ $\cancel{Ext^k_R}$ $Ext^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, Q) \cong Q/nQ$.

Recall the statement of Universal coefficient thm:

Thm 3.2 Fix R (PID), $Q \in \underline{\text{Mod}}_R$, $C_* \in \underline{\text{Kom}}_R$ of <u>free</u> R modules. Then:

$$0 \to \text{Ext}^1_R(H_{k-1}(C), Q) \to H^k(C, Q) \xrightarrow{h^k} \text{Hom}_R(H_k(C), Q) \to 0$$

for all k is a <u>split</u> short exact sequence.

<u>Also</u>: the splitting may be "unnatural" but the sequence is natural.

Clarification) $UCT: \underline{\text{Kom}}^{\text{free}}_R \longrightarrow \underline{\text{SES}}\,\underline{\text{Grad}}^R$, that is if $f: C_* \to D_*$ then $UCT(f)$ is a chain map from $UCT(D)$ to $UCT(C)$

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Pf Idea: ① Define h^*
② Proove h well-defined
③ h is R-linear
④ h is surjective
⑤ understand $\ker(h)$.

So:

① we want a definition of h. Fix $\varphi \in C^*_k$. (This is $\text{Hom}(C_k, G)$).

Fix $z \in Z_k (= \ker(\partial_k))$

Define : $h([\varphi])[z] = \varphi(z)$

Alternatively: (Saul says this is better)

Fix $[\varphi] \in H^k(C,Q)$. Fix $[z] \in H_k(C)$.

That is $\varphi \in Z^k$, $z \in Z_k$. Define

$$h([\varphi])([z]) = \varphi(z)$$

~~ok~~ to check: (i) Domain
(ii) Codomain
(iii) representatives.

(i) ~~Doma~~ Domain is correct, it is defined on $H^*(C,Q)$.

(ii) Is $h([\varphi])$ an R-linear homomorphism?

so, fix $[w] \in H_k(C)$ and $r, s \in R$ then

$$\begin{aligned} h([\varphi])(r[z]+s[w]) &= h([\varphi])([rz+sw]) \\ &= \varphi(rz+sw) \\ &= r\varphi(z)+s\varphi(w) \\ &= rh([\varphi])[z]+sh([\varphi])[w], \end{aligned}$$

so $h([\varphi])$ is R-linear and the codomain is correct. [illegible] (iii), or (2) above. Suppose $\varphi + \delta\psi \in [\varphi]$ and

$z + \partial c \in [z]$.

Then $$h([\varphi+\delta\psi])([z+\partial c]) = (\varphi+\delta\psi)(z+\partial c)$$
$$= \varphi(z) + \varphi(\partial c) + \delta\psi(z) + \delta\psi(\partial c)$$
$$= \varphi(z) + \underbrace{\cancel{\delta\varphi(c)}}_{0} + \underbrace{\cancel{\psi(\partial z)}}_{0} + \underbrace{\psi(\cancel{\partial^2} c)}_{0\text{, as } \partial^2 = 0}$$
$$= \varphi(z).$$

③ h is R-linear. To show this:

$$h(r[\varphi] + s[\theta])([z]) = h([r\varphi + s\theta])([z])$$
$$= (r\varphi + s\theta)(z)$$
$$= r\varphi(z) + s\theta(z)$$
$$= (rh([\varphi]) + sh([\theta]))[z].$$

④ Surjectivity. Fix $\theta \in Hom(H_k(C), Q)$,

so $\theta : Z_k/B_k \to Q$. Let $q_k : Z_k \to Z_k/B_k$

be the quotient. Consider ~~$0 \to Z_k \to C_k$~~

$$0 \to Z_k \xrightarrow{i} C_k \to \text{image}(\partial_k) \to 0$$

· Since $\text{image}(\partial_{k+1}) < C_{k-1}$, we have $\text{image}(\partial_k)$ is free and SES splits. Fix a projection ← we use that R is a PID here.

$$P_k : C_k \to Z_k.$$

· Consider $\theta \circ q \circ p : C_k \to Q$

Exercise (a) $\delta(\theta \circ q \circ p) = 0$, so $\theta \circ q \circ p \in Z^k$

(b) $h([\theta \circ q \circ p]) = \theta$

Note: Saul uses ϕ (varphi) instead of θ in his lecture