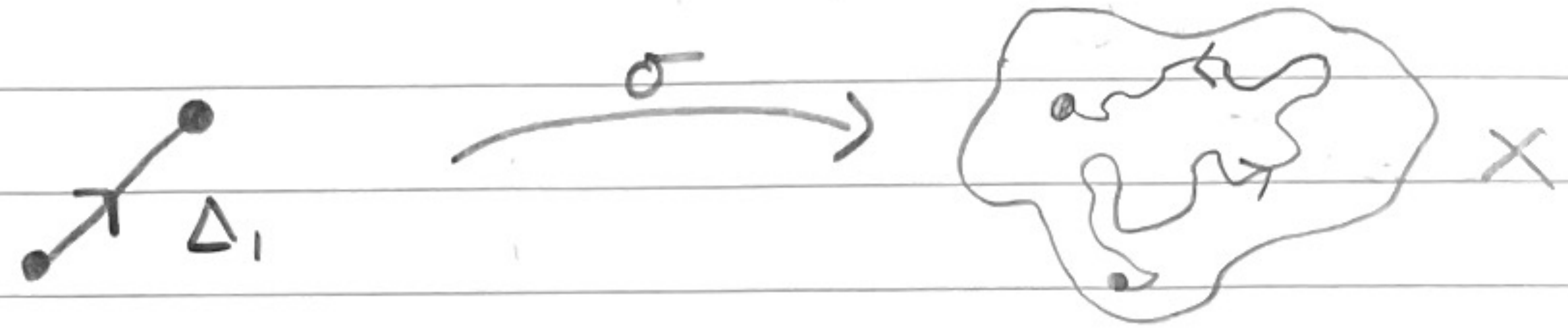
Cohomology 9

The Winding cochain

X - topological Space

$\Delta^n = \{x \in \mathbb{R}^{n+1} : \sum x_i = 1, x_i \geq 0\}$

A singular n-simplex in X is a continuous map $\sigma^n : \Delta^n \to X$



Define $C_n^{sing}(X;R)$ to be the free R-module generated by singular n simplices

Define $\partial_n \sigma_n = \sum (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$

exercise

$\partial_{n-1} \circ \partial_n = 0$

So $C_*^{sing}(X;R) = (C_k^{sing}(X;R))_{k\in\mathbb{Z}}$ is a chain complex.

Define $Z_k = \mathrm{Ker}(\partial_k)$ - Cycles
$B_k = \mathrm{Im}(\partial_{k+1})$ - Boundaries
$H_k = Z_k / B_k$

Example

$$H_k(S^1;\mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } k=0,1 \\ 0 & \text{else} \end{cases}$$

Likewise if $\mathbb{R} = \mathbb{Z}$

Define $C^k_{sing}(X;Q) = \mathrm{Hom}_R(C^{sing}_k(X,R); Q)$
Define $\delta^k = (\partial_k)^*$ Then $C^*_{sing}(X;Q) = (C^k_{sing}, \delta^{k+1})_{k \in \mathbb{Z}}$

$C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1}$

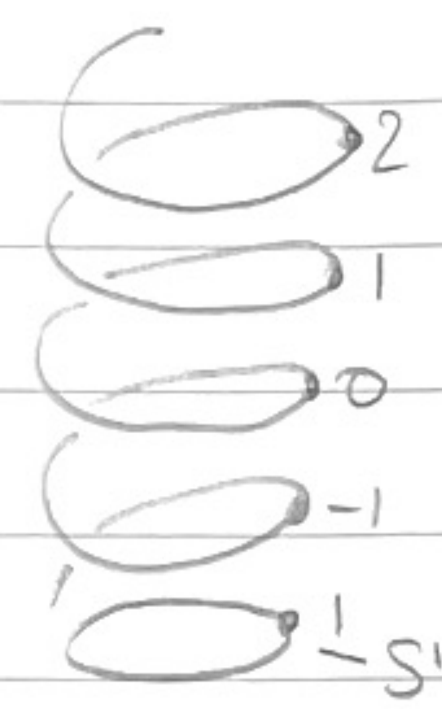
Dualise:

$C^{k+1} \xleftarrow{\delta^{k+1}} C^k \xleftarrow{\delta^k} C^{k-1}$

and $Z^k = \mathrm{Ker}(\delta^{k+1})$ - cocycles
$B^k = \mathrm{Im}(\delta^k)$ - coboundaries
$H^k = Z^k / B^k$ - cohomology

The Winding Cocycle

Define $\exp : \mathbb{R} \to S^1$
$t \mapsto \exp(2\pi i t)$



Suppose $\sigma : \Delta \to S^1$ is a singular simplex.
Define $\tilde{\sigma} : \Delta^1 \to \mathbb{R}$ to be (any) Path lift of σ

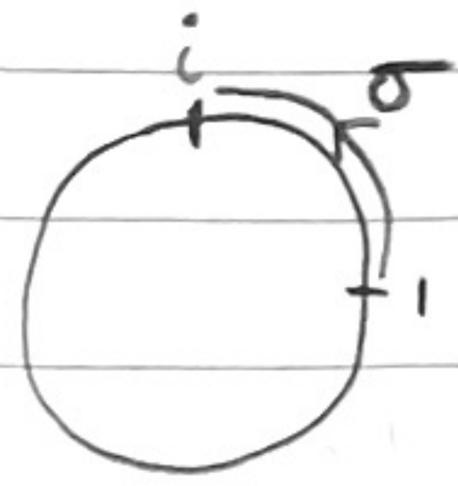
$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\exp"] \\ \Delta^1 \arrow[ur, dashed, "\tilde{\sigma}"] \arrow[r, "\sigma"'] & S^1 \end{tikzcd}$$

that is: $\sigma = \exp \circ \tilde{\sigma}$

Define $\omega(\sigma) = \tilde{\sigma}(1) - \tilde{\sigma}(0)$ extend R-linearly to get $\omega : C^{sing}_1(S^1; R) \longrightarrow \mathbb{R}$

Exercises

① ω is well defined (independent of choice of $\tilde{\sigma}$)
② $\omega \notin B^1_{sing}(S^1;\mathbb{R})$
③ $\omega \in Z^1_{sing}(S^1;\mathbb{R})$ $\quad(\delta^2\omega = 0)$
④ $H^1_{sing}(S^1;\mathbb{R}) \cong \mathbb{R}$ is generated by ω



winding of σ is $\frac{1}{4}$
$\omega(\sigma) = \frac{1}{4}$
So ω takes values in $\mathbb{R}$

Exercise

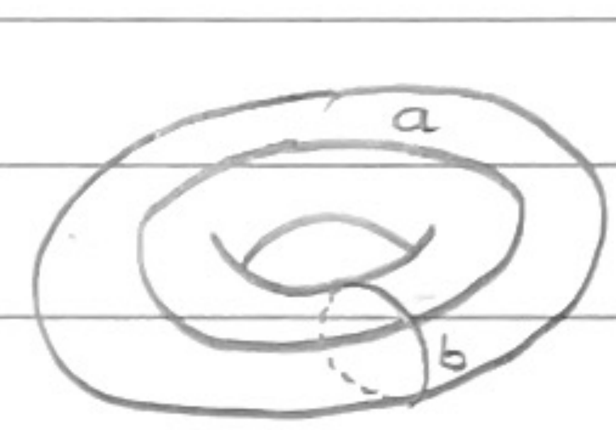
Give a direct proof that $H^1_{sing}(S^1;\mathbb{Z}) \cong \mathbb{Z}$
Also describe the generator.

Remark

Of course $H^1_{sing}(S^1;\mathbb{Z}) \cong \mathbb{Z}$ following from the universal coefficient theorem because $\mathrm{Ext}^1(H_0(S^1);\mathbb{Z}) \cong 0$

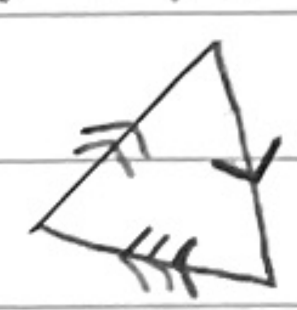
Example

Define $\mathbb{T}^2 = S^1 \times S^1$. Here there are two winding cocycles "parallel to a or b"
Again $H^1_{sing}(\mathbb{T}^2;\mathbb{R}) \cong \mathbb{R}^2$ generated by these.



Q: What is δ^1?
Suppose $\varphi: C_1 \to Q$, Suppose $c \in C_2$ then
$(\delta\varphi)(c) = \varphi(\partial c)$ by definition

c



Q Suppose $\tilde{X}$ is "nice" and Γ is a group acting nicely on $\tilde{X}$. Can we understand the cohomology $H^*(\tilde{X}/\Gamma)$ in terms of $H^*(\tilde{X})$ and Γ
[Group cohomology]

20. Reduced cohomology

Suppose $C_* = C_*^{sing}(X, R)$

Define $\tilde{C}_* = (\tilde{C}_k, \tilde{\partial}_k)$ by

$$\tilde{C}_k = \begin{cases} C_k & \text{if } k \neq -1 \\ R & \text{if } k = -1 \end{cases}$$

$$\tilde{\partial}_k = \begin{cases} \partial_k & \text{if } k \neq 0 \\ \varepsilon & \text{if } k = 0 \end{cases}$$

where ε is the augmentation homomorphism

$$\varepsilon : C_0(X;R) \longrightarrow R$$
$$\sigma_0 \longmapsto 1_R$$

Define $\tilde{H}^k_{sing}(X;R) = H^k(\tilde{C})$

Exercise

$$\tilde{H}^k \cong \begin{cases} H^k & \text{if } k \geq 1 \\ H^0/R & \text{if } k = 0 \\ 0 & \text{if } k \leq -1 \end{cases}$$

Hence $H^0 \cong \{\text{functions from path components to } R\}$
$H^0/R = H^0/\text{constant functions}$

21. Relative Cohomology

Suppose $B \subset A \subset X$ are spaces. We call (X, A, B) a triple. We define $C_*(X,A) := C_*(X)/C_*(A)$ for a pair (X,A). Similarly define $C_*(X,B)$, $C_*(A,B)$

Check

$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X,A) \longrightarrow 0$

and

$0 \longrightarrow C_*(A,B) \longrightarrow C_*(X,B) \longrightarrow C_*(X,A) \longrightarrow 0$

are split, short exact sequences of chain complexes

So we may dualise to get

$0 \longleftarrow C^*(A,B) \longleftarrow C^*(X,B) \longleftarrow C^*(X,A) \longleftarrow 0$

this gives a long exact sequence of relative cohomology.