LECTURE 10

2022-01-31

(21) Relative Cohomology

Suppose $B \overset{j}{\subseteq} A \overset{i}{\subseteq} X$ are spaces

Call (X, A, B) a triple of spaces

There is a short exact sequence of chain complexes

$$0 \to C_*(A,B) \xrightarrow{i} C_*(X,B) \xrightarrow{q} C_*(X,A) \to 0$$

Which splits (exercise)

Thus, dualising, we again obtain a short exact sequence of cochain complexes and thus a long exact sequence in cohomology

$$\begin{tikzcd} H^*(A,B) \arrow[dr, "\delta" ', "{[+1]}"] & & H^*(X,B) \arrow[ll, "i^*" '] \\ & H^*(X,A) \arrow[ur, "q^*" '] & \end{tikzcd}$$

It is more common to take $B = \emptyset$ and, writing

$$H^*(X, \emptyset) = H^*(X)$$

We have the LES (overleaf):

$$\begin{tikzcd} H^*(A) \arrow[dr, "\delta"'] & & H^*(X) \arrow[ll, "i^*"'] \\ & H^*(X,A) \arrow[ur, "q^*"'] & \end{tikzcd}$$

$[+1]$

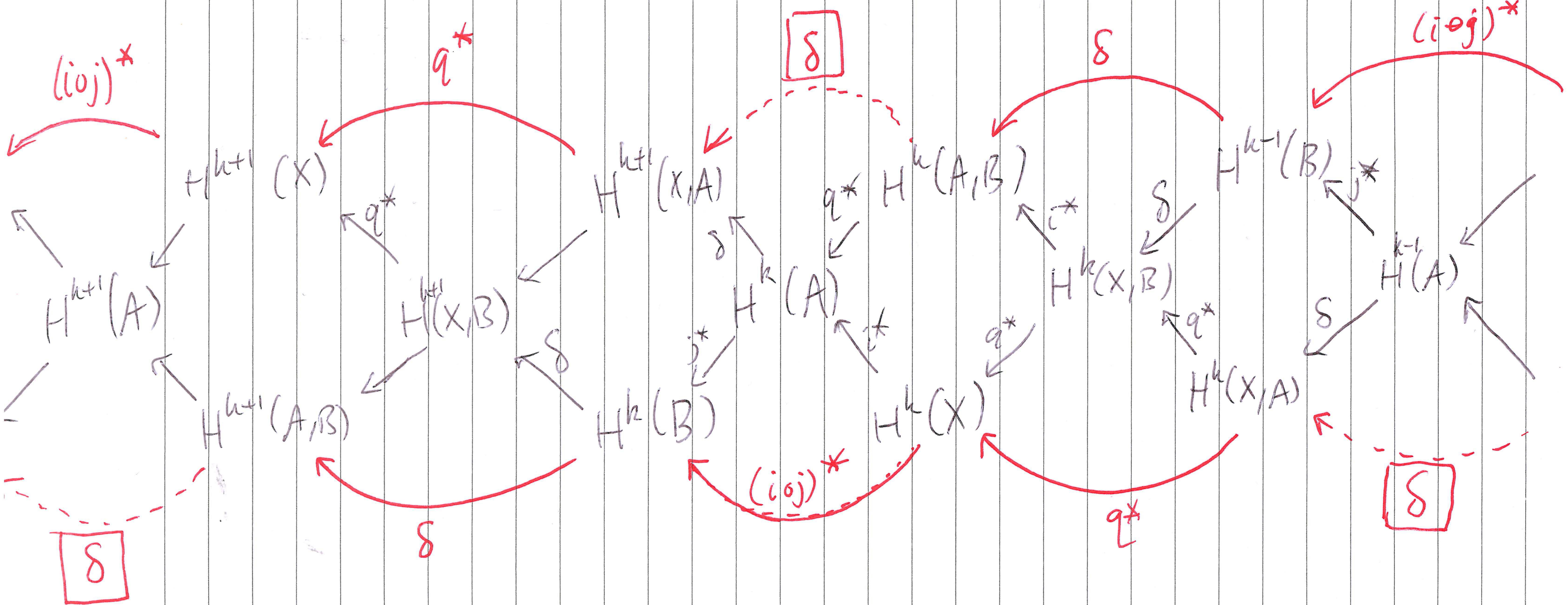
Exact Triangle

This exact triangle is an abuse of notation and represents a LES. The $[+1]$ indicates where the degree shift happens.

Since (X,A,B) is a triple, we have three pairs, namely (X,A), (X,B), (A,B)

Each gives a LES in cohomology.

These may be used to give the braid diagram


(i∘j)*
H^{k+1}(X)
q*
H^{k+1}(X,A)
δ
H^k(A,B)
δ
H^{k-1}(B)
(i∘j)*
H^{k+1}(A)
H^{k+1}(X,B)
H^k(A)
H^k(X,B)
H^{k-1}(A)
H^{k+1}(A,B)
H^k(B)
H^k(X)
H^k(X,A)
i*
j*
δ
(i∘j)*
q*
δ
δ

Theorem: (Braid Lemma) The exactness of the LES of the triple (X, A, B) can be deduced from the long exact sequences of the three pairs and the braid diagram.

This, therefore, is another way to prove that we have a LES of the triple.

QUESTION: What is the connecting homomorphism (on cochains?)

Recall: The connecting homomorphism

$$\partial_k : H_k(X, A) \to H_{k-1}(A)$$

has a topological picture.

Fix $[[z]] \in H_k(X, A)$. So

$$[z] \in Z_k(X, A) = \cancel{\ker \{ c \in C_k(X) \mid \partial c \in C_{k-1}(A) \}}$$

$$\subseteq C_k(X, A) = C_k(X) / C_k(A)$$

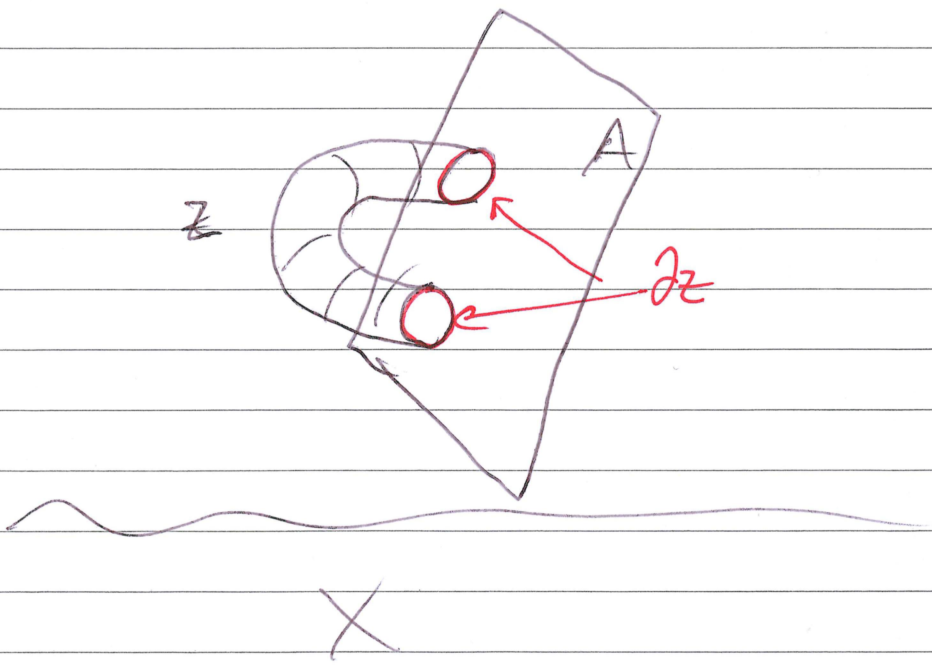
$z \in C_k(X)$

Then [Note $Z_k(X,A) = \ker(\partial_k^{(X,A)})$

$$\partial_k^{(X,A)} : C_k(X,A) \to C_{k-1}(X,A)$$

Thus, $\partial z \in C_{k-1}(A)$

<u>Picture</u>:



Note ∂z is a cycle in $C_{k-1}(A)$ because

$\partial^2 z = 0$

So, $\partial_k [[z]] = [\partial z]$

Lies in $H_{k-1}(A)$

Moral: It would be nice if $\delta = (\partial)^*$

But it is sadly not true.

Instead, we have

Lemma:

$$\begin{tikzcd} H^k(X,A) \arrow[d, "h_k"'] & H^{k-1}(A) \arrow[l, "\delta^k"'] \arrow[d, "h_{k-1}"] \arrow[dl, phantom, "\circ"] \\ \mathrm{Hom}(H_k(X,A)) & \mathrm{Hom}(H_{k-1}(A)) \arrow[l, "(\partial_k)^*"] \end{tikzcd}$$

The diagram commutes

So, $h \circ \delta = \partial^* \circ h$

Pf: Exercise $\square$

Hatcher says:

$$C^k(X,A) = \mathrm{Hom}(C_k(X,A), \mathbb{Q})$$

is "complicated"

Set $i : A \to X$ inclusion

Induces $i_k : C_k(A) \to C_k(X)$

and $i^k : C^k(A) \leftarrow C^k(X)$

Lemma: $C^k(X, A)$ is naturally isomorphic (via q^*) to

$\ker(i^*)$

$= \{ \phi \in C^k(X) \mid \phi(c) = 0 \text{ for } c \in C_k(A) \}$

Again, here $q_k : C_k(X) \to C_k(X, A)$

and q^k is the dual.