Lec 11 2022-02-01 (WANYING ZHAO)

(22) Excision

$\boxed{1}$ Five Lemma: suppose A, B, C, D, E, A', B', C', D', E' are R-modules
$\alpha, \beta, \gamma, \delta, \varepsilon$ are morphisms (eg. $\alpha: A \to A'$)

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \downarrow\varepsilon \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

has exact rows and squares comute and $\alpha, \beta, \delta, \varepsilon$ are iso

Then γ is an isomorphism.

[Ex 1] Do diagram chasing

[Ex 2] Show you can weaker the hypothesis on α, ε.

$\boxed{2}$ Excision I: Suppose $Z \subseteq A \subseteq X$ are spaces with closure$(Z) \subseteq$ interior(A)

Define $Y = X - Z$ $B = A - Z$ $i: (Y, B) \hookrightarrow (X, A)$

Then Thm: $i_*: H_*^{sing}(Y, B) \to H_*^{sing}(X, A)$ is ISO.

$i^*: H^*_{sing}(X, A) \to H^*_{sing}(Y, B)$ is ISO

Excision II: Suppose $A, B \subseteq X$ with $X \subseteq \operatorname{int}(A) \cup \operatorname{int}(B)$

Then $H_*(B, B\cap A) \cong H_*(X, A)$

Excision III: Suppose X is CW and $A, B \subseteq X$ are subcomplexes with
$X \subseteq A \cup B$

Then $H_*(B, B\cap A) \cong H_*(X, A)$

Proof

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathrm{Ext}^1_R(H_{*-1}(X,A),Q) \arrow[r] \arrow[d, "\mathrm{Ext}(i_{*-1})", "\cong \ (A)"'] & H^*(X,A;Q) \arrow[r, "h(X,A)"] \arrow[d, "i^*"] & \mathrm{Hom}_R(H_*(X,A),Q) \arrow[r] \arrow[d, "(i_*)^*", "\cong \ (B)"'] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Ext}^1_R(H_{*-1}(Y,B),Q) \arrow[r] & H^*(Y,B;Q) \arrow[r, "h(Y,B)"] & \mathrm{Hom}_R(H_*(Y,B);Q) \arrow[r] & 0
\end{tikzcd}$$

(A) is iso b/c Ext^1 is a functor and by Excision for H_*

(B) is iso ... Hom ...

[Ex3]: Use LES for a pair

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}_R(H_{k-1}(A)) \arrow[r] & H^k(A) \arrow[r] & \mathrm{Hom}(H_k(A)) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}_R(H_{k-1}(X)) \arrow[r] \arrow[u] & H^k(X) \arrow[r] \arrow[u] & \mathrm{Hom}(H_k(X)) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{Ext}_R(H_{k-1}(X,A)) \arrow[r] \arrow[u] & H^k(X,A) \arrow[r] \arrow[u] & \mathrm{Hom}(H_k(X,A)) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{Ext}_R(H_{k-2}(A)) \arrow[r] \arrow[u] & H^{k-1}(A) \arrow[r] \arrow[u] & \mathrm{Hom}(H_{k-1}(A)) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

[Ex4] Prve that $H^*_\Delta \cong H^*_{sing}$

(23) Axioms for cohomology

$\boxed{\text{Def}}$ $\underline{\text{Pairs}_{CW}}$:= cat with obj (X, A) X: CW complex, A sub complex
morphism: cts maps of pairs (*)

$\boxed{\text{A cohomology theory}}$ is

(i) A functor (contra var) $h: \underline{\text{Pairs}}_{CW} \longrightarrow \underline{\text{Grad}}^R$ with notation $h(A, \emptyset) := h(A)$

(ii) A natural transformation

$$\begin{array}{ccc} (X, A) & \longrightarrow & h^{*-1}(A) \\ & \Downarrow \delta & \\ (X, A) & \longrightarrow & h^*(X, A) \end{array}$$

which satisfies the axioms:

(I) [Homotopy] if $f \simeq g : (X, A) \to (Y, B)$ Then $h(f) = h(g)$

(II) [Excision] $A, B \le X$ sub cplx, $X = A \cup B$. Then $h(i)$ is iso.
$h(X, A) \cong h(B, B \cap A)$

(III) [LES] Suppose (X, A) is a CW pair. So $(A, \emptyset) \xhookrightarrow{i} (X, \emptyset)$
$(X, \emptyset) \xhookrightarrow{j} (X, A)$

Then $\exists$ LES

$$\cdots \to h^*(A) \xleftarrow{i^*} h^*(X) \xleftarrow{j^*} h^*(X, A) \xrightarrow{\delta} h^{*-1}(A) \leftarrow \cdots$$

[IV] [Disjointness]

Suppose $(X, A) = \coprod_{\alpha}(X_\alpha, A_\alpha)$ with $i_\alpha : (X_\alpha, A_\alpha) \hookrightarrow (X, A)$

Then $\prod i_\alpha^* : h(X, A) \longrightarrow \prod_\alpha h(X_\alpha, A_\alpha)$ is an iso.