Lecture 12

Thursday, 3 February 2022 12:06 PM

Note: The LES for a triple (X, A, B) on be induced from the axioms.

Note: We gave four axioms for unreduced cohomology theories
Hatcher (p202) gives three axioms which define reduced cohomology theories.

Exercise: $T : \mathbf{Ring}_{cm} \to \mathbf{Grad}^{ab}$

$(X, A) \mapsto 0$ is a cohomology theory

Exercise: H^n_{sing} is an (unreduced) cohomology theory

By theorem $\left\{ \begin{array}{l} \\ \\ \\ \\ \end{array} \right.$, we have $\left\{ \begin{array}{l} \text{axiom} \\ \text{axiom} \end{array} \right.$

Proposition: Suppose h, k are cohomology theories.
Suppose $\mu : h \to k$ is a natural transformation st
$\mu_{(pt,\, \emptyset)} : h(pt, \emptyset) \to k(pt, \emptyset)$ is an isomorphism
Then μ is a natural isomorphism.

Proof: SEE lecture 20?

"Yeah I wrote sorry it but I can't say it grumble grumble grumble..."

Later, we could use this to prove $H^n_{cw} \cong H^n_{sing}$

Eg: There is a natural isomorphism from H^1 to H^1_{sing} "Why did I do this twice that's so weird"

Cellular cohomology

Recall a CW-complex X is an ascending union (closure finite weak topology)
$X = \bigcup_{n \geq 0} X^{(n)}$ where $X^{(n)}$ is the k-skeleton of X.

$k = 0$: $X^{(0)}$ is a collection of points (0-cells) with discrete topology.

$k+1$: Given $X^{(k)}$, obtain $X^{(k+1)}$ as a pushout
$\begin{array}{ccc} \coprod_\alpha S^k_\alpha & \to & X^{(k)} \\ \downarrow & & \downarrow \\ \coprod_\alpha D^{k+1}_\alpha & \to & X^{(k+1)} \end{array}$
where $D^{k+1}_\alpha \cong B^{k+1}$ closed ball $\forall \alpha$.
and we attach D^{k+1}_α to $X^{(k)}$ via a map $\varphi_\alpha : S^k_\alpha \to X^{(k)}$

So $X^{(k+1)} = \frac{X^{(k)} \amalg (\coprod_\alpha D^{k+1}_\alpha)}{x \sim \varphi(x)}$

[Due to JHC Whitehead.]
Joint his contrib.

C = Closure finite, every compact subspace meets only finitely many open cells $(D^n_\alpha)^\circ$
W = Weak topology, $A \subseteq X$ open iff $A \cap X^{(n)}$ is open for all k.

Exercise: Find a CW structure on S^n with minimal number of cells

Exercise: Find a CW structure on $\mathbb{RP}^n$ with minimal number of cells

Exercise: Find a CW structure on $T^1 = \bigvee_{n=1} S^1$ the n torus, minimal number of cells.

Exercise: Find a CW structure on $\mathbb{CP}^n$ with minimal number of cells

Question: Can you skip a dimension in CW construction?

Eg: Yes! Take $X = S^2$

$X^{(0)} = \{N\}$
$X^{(1)} = X^{(0)}$
$X^{(2)} = X^{(1)} \amalg D^2 / \sim$ where attaching map

Here, no one cells

Question: Do the same axioms work for cohomology theories on other categories?
i.e. not $\mathbf{Pairs}_{cw}$?

Answer: Saul doesn't know. "Certainly a thing you could go and think about probably in a graduate course in algebraic geometry."

All of M is a nice manifold. Then it admits a CW-structure

Important note: If X is CW then
$X^{(n)}/X^{(n-1)} \cong \bigvee_{\text{wedge}} S^n$
Where the wedge product over one copy over all k cells of $X^{(k)}$

Eg:

0 cell : v
1 cells : a, b, c
2 cells : U, L

$X^0 \cong pt$
$X^1/X^0 \cong \bigvee_{a,b,c} S^1$
$X^2/X^1 \cong \bigvee_{U,L} S^2$

Homology

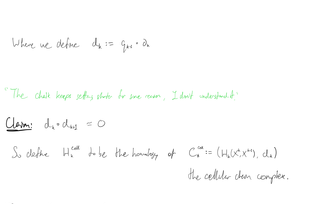
Recall for X a CW complex,

$H_n(X^k, X^{k-1}) \cong \left\{ \begin{array}{ll} \mathbb{Z}^{\text{\#}k\text{-cells}} & n = k \\ 0 & \text{else} \end{array} \right.$

Note: if $k=0$
$H_n(X^0, \emptyset) = \mathbb{Z}$
Works in all dimensions

Where the sum is over all the k-cells of X.

"Look at all the spaces... Let's fill it with a diagram"



Where we define $d_{ij} := q_{ij} \circ \partial_i$

"The chain keeps getting shorter for some reason. I don't understand it"

Claim: $d_{ij} \circ d_{jk} = 0$

So define H^{cw}_n to be the homology of $C^{cw}_n := (H_n(X^n, X^{n-1}), d_n)$
the cellular chain complex.

This was the rationale for chain complexes

D^2 : $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$

$\downarrow$ 2 cells

$0 \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to 0$

Problem: Two reasonable constructions of H^n_{cw}

1. $(C^{cw}_\bullet)^*$ then take H^*
OR
2. take $H^*(X^*, X^{*-1})$ deducing earlier to get a corresponding diagram of cohomology.

"Come back next week for the exciting resolution!"

Ask Sunny lots of questions. Don't let him lecture.
No new information!