# Cohomology + PD

## 24 Cellular cohomology

Suppose $X$ is CW.

$$C^k_{cell}(X) = \mathrm{Hom}_R(H_k(X^k, X^{k-1}), Q)$$

$$C^*_{cell}(X) = (C^k_{cell}(X), (d_k)^*)$$

Define $H^*_{cell}$ via the cohomology of $C^*_{cell}$

Def 2: Consider the LES of cohomology of pairs $(X^k, X^{k-1})$ and make the 'same' definition:

$$\begin{tikzcd} H^k(X^k, X^{k-1}) & & H^{k-1}(X^{k-1}, X^{k-2}) \arrow[ll, "\delta^k"'] \arrow[dl, "q^*"] \\ & H^{k-1}(X^{k-1}) \arrow[ul, "\delta_k"] & \end{tikzcd}$$

Define

$$D^*_{cell}(X) = (H^k(X^k, X^{k-1}), d^k)$$

and define $\hat{H}^*_{cell}$ to be the cohomology.

Theorem 3.5: The homomorphism $h : H^* \to \mathrm{Hom}(\cdot, Q)$ gives an isomorphism between $\hat{H}^*_{cell}$ and $H^*_{cell}$.

Furthermore, $H^*_{cell} \cong H^*_{sing}$ (†) (natural isomorphism) (as functors $\underline{\mathrm{PairsCW}} \to \underline{\mathrm{Grd}^R}$)

Proof (furthermore):
Use the UCT and the five lemma.

Proof of 3.5: A medium-sized diagram:

$$\begin{tikzcd}
H^k(X^k, X^k; Q) \arrow[d, "h"] \arrow[dr, phantom, "\circ"] & H^{k-1}(X^{k-1}; Q) \arrow[l, "\delta^k"'] \arrow[d, "h"] \arrow[dr, phantom, "\circ"] & H^{k-1}(X^{k-1}, X^{k-2}; Q) \arrow[l, "q^{k-1}"'] \arrow[d, "h"] \\
\mathrm{Hom}(H_k(X^k, X^{k-1}), Q) & \mathrm{Hom}(X_{k-1}(X^{k-1}); Q) \arrow[l, "(\partial_k)^*"'] & \mathrm{Hom}(H_k(X_{k-1}, X_{k-2}), Q) \arrow[l, "(q_{k-1})^*"]
\end{tikzcd}$$

- each square here commutes (see previous lectures)

Thus also $H_{k-1}(X^k, X^{k-1}) \cong 0$ (by excision), thus the outer $h$ maps are isomorphisms since $\mathrm{Ext}^1_R(0, Q) = 0$ $\square$ (3.5)

Q: Pairs$_{CW}$ $\subset$ Pairs? So $H^*_{cell}$ only exists for the former?

A: Yes. Challenge: Suppose $X \subset \mathbb{R}^2$ is path connected. Prove $H_k(X, \mathbb{Z}) \cong 0$ $\forall k \geq 3$.

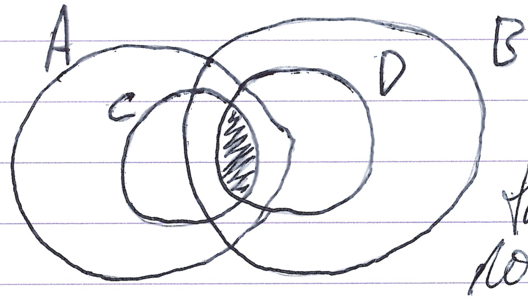
eg. $\mathbb{R}^2 \setminus \{\text{Cantor set}\}$

Moral of (†), singular is good for proving theorems, but cellular is better for computation.

(25) Mayer-Vietoris:
Suppose $(X, Y)$ is a pair, and $(A, C), (B, D) \subset (X, Y)$ are pairs with

$$X \subset \mathrm{int}(A) \cup \mathrm{int}(B) \qquad \text{(interior)}$$
$$Y \subset \mathrm{int}(C) \cup \mathrm{int}(D)$$

Picture:



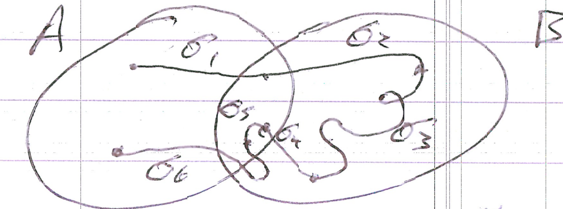

Note if $C = D = Y = \emptyset$ then we have the normal M.V. LES.

Definition: We call $\mathcal{U} = \{(A,C),(B,D)\}$ an 'excisive cover' of $(X,Y)$

Def: A chain $c = \sum_\alpha r_\alpha \sigma_\alpha$ is subordinate to $\mathcal{U}$ if for each $\alpha$, $\sigma_\alpha$ lies in one of the two elements of the cover.

So: $C_k^{\mathcal{U}}(X)$ = chains subordinate to $\{A,B\}$

$C_k^{\mathcal{U}}(Y)$ = chains subordinate to $\{C,D\}$




Thus $C_k^{\mathcal{U}}(Y) \hookrightarrow C_k^{\mathcal{U}}(X)$ is an inclusion

Define $C_k^{\mathcal{U}}(X,Y)$ to be the cokernel

Thus

$$0 \to C_*^{\mathcal{U}}(Y) \to C_*^{\mathcal{U}}(X) \to C_*^{\mathcal{U}}(X,Y) \to 0$$

is a short exact split sequence (exercise)

In the proof of excision ($2^{nd}$ version), we showed

$$C_*(X) \simeq C_*^{\mathcal{U}}(X)$$

inducing an isomorphism on $H_* \cong H_*^{\mathcal{U}}$

Similarly

$$\begin{tikzcd}
0 \arrow[r] & C_*(Y) \arrow[r] & C_*(X) \arrow[r] & C_*(X,Y) \arrow[r] & 0 \\
0 \arrow[r] & C_*^{\mathcal{U}}(Y) \arrow[r] \arrow[u, hook] & C_*^{\mathcal{U}}(X) \arrow[r] \arrow[u, hook] & C_*^{\mathcal{U}}(X,Y) \arrow[r] \arrow[u, "\circledast"'] & 0
\end{tikzcd}$$

- All inclusions are chain homotopy equivalences.

By 5ive lemma and LES. deduce that ⊛ gives isom $H^{u}_{*}(X,Y) \cong H_*(X,Y)$

NOTE: ⊛ is in fact not an inclusion.

Claim: Both rows above are split, thus we may dualise and prove $H^*_{u}(X,Y;\mathbb{Q}) \cong H^*(X,Y;\mathbb{Q})$