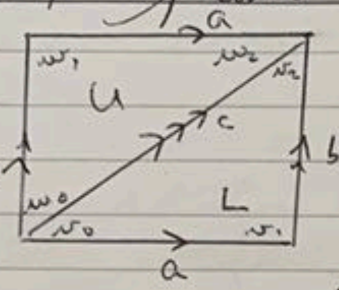
Lecture 15 - 10/2/2022

(26) Cup product



→ cell structure on $\mathbb{T}^2$.

- note: $\partial L = b - c + a$,
  & $b = L|_{[v_1, v_2]}$,
  $= U|_{[v_0, v_1]}$,

and $a|_{[v_0]} = v$

- Thus, C_*^{cell} is
  $$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z} \longrightarrow 0$$
  $$\langle U, L\rangle \quad \langle a, b, c\rangle \quad \langle v\rangle$$
- Hence C^*_{cell} is
  $$0 \longleftarrow \mathbb{Z}^2 \longleftarrow \mathbb{Z}^3 \longleftarrow \mathbb{Z} \longleftarrow 0$$
  $$\langle U^*, L^*\rangle \ \langle a^*, b^*, c^*\rangle \ \langle v^*\rangle$$
- Here, $a^*(a) = 1$, and $a^*(b) = a^*(c) = 0$.
- note: $\partial(L - U) = 0$; $[\partial U = a - c + b)$.
- Define $z = L - U$ - this class $[z]$ generates $H_2(\mathbb{T}^2; \mathbb{Z})$.

- Exercise: Give generators for H^k, for $k = 0, 1, 2$.
- Define (over $\mathbb{R}$), $\alpha, \beta \in Z^1(\mathbb{T}^2; \mathbb{R})$.

- We computed (last time)
  $$(\alpha \smile \beta)(z) = 1, \text{ and } (\beta \smile \alpha)(z) = -1,$$
  so we'd suspect the operation's anticommutative.
- note: $\delta(\alpha \smile \beta) = 0$, and
  claim: $[\alpha \smile \beta] \in H^2(\mathbb{T}^2; \mathbb{R})$ is a (nonzero) generator.

- Exercises: (1) The cup product on cochains is $\mathbb{R}$-bilinear (linear in both coordinates).
  (2) The cup product is associative.
  (3) $\varepsilon : C_0(X) \to \mathbb{R}$,
  $\sigma^0 \mapsto 1_{\mathbb{R}}$, is a multiplicative identity for the cup product.

- Lemma 3.6: (Leibniz Rule) - suppose $\phi \in C^k$, $\psi \in C^l$.

Then $\delta(\phi \smile \psi) = (\delta\phi) \smile \psi + (-1)^k \phi \smile (\delta \psi)$.

- That is, this is the Leibniz rule at the level of cochains.
- Pf: $\delta(\phi \smile \psi) \in C^{k+l+1}(X)$

  set $N := k+l+1$.

- Suppose $\sigma : \Delta^n \to X$ - singular simplex.

$\rightsquigarrow \partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_N]}$.

- $A = \sum_{i=0}^{k} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_k, v_{k+1}, \dots, v_N]}$.
- $B = \sum_{i=k+1}^{N} (-1)^i \sigma|_{[v_0, \dots, v_k, v_{k+1}, \dots, \hat{v}_i, \dots, v_N]}$.
- note: $\partial\sigma = A + B$.
- Define $s = \sigma|_{[v_0, \dots, v_k]} = \sigma|_{[v_0, \dots, v_k, \hat{v}_{k+1}]}$,

  and $t = \sigma|_{[v_{k+1}, \dots, v_N]} = \sigma|_{[\hat{v}_k, v_{k+1}, \dots, v_N]}$.

- Define $A' = \sum_{i=0}^{k} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_k, v_{k+1}]}$ } restricted to front faces

  and $B' = \sum_{i=k+1}^{N} (-1)^i \sigma|_{[v_k, \dots, \hat{v}_i, \dots, v_N]}$ } restricted to back faces.

- note A' is missing a term,

  namely $(-1)^{k+1} \sigma|_{[v_1, \dots, v_k, \hat{v}_{k+1}]} = (-1)^{k+1} \cdot s$.

- B' is missing $(-1)^k \sigma|_{[\hat{v}_k, v_{k+1}, \dots, v_N]} = (-1)^k \cdot t$.
- Thus, $\partial(\sigma|_{[v_0, \dots, v_{k+1}]}) = A' + (-1)^{k+1} s$,

  and $\partial(\sigma|_{[v_k, \dots, v_N]}) = (-1)^k \cdot B' = (-1)^k((-1)^k \cdot t + B')$.

- Finally, compute ~~$\delta(\phi\smile\psi)(\sigma) = (\phi \smile\psi)\delta(\sigma)$~~

  $\delta(\phi \smile \psi)(\sigma) = (\phi \smile \psi)(\partial\sigma)$, (by definition of coboundary operator δ).

- $(\phi \smile \psi)(\partial\sigma) = (\phi \smile \psi)(A+B)$ (by definition of A, B).
- $(\phi \smile \psi)(A+B) = (\phi \smile \psi)(A) + (\phi \smile \psi)(B)$, by distributivity, and by extending the cup product linearly.

k+1 terms, l+1 terms

- $(\phi\smile\psi)(A)+(\phi\smile\psi)(B)$

$$= (\phi\smile\psi)\left(\sum(-1)^i\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_k,v_{k+1},\dots,v_N]}\right)$$

$+(\phi\smile\psi)(B)$, by definition of A.

- They overlap on the $(k+1)$th term, so we get:

$\dots = \phi(A')\cdot\psi(t)+\phi(s)\cdot\psi(B')$, by the definition of A', B', s, t.

adding ~~adding~~ and subtracting the same thing

$$\dots = \phi(A')\cdot\psi(t) + (-1)^{k+1}\phi(s)\cdot\psi(t) + (-1)^k\phi(s)\cdot\psi(t) + \phi(s)\cdot\psi(B').$$

$$= \left(\phi(A') + (-1)^{k+1}\phi(s)\right)\cdot\psi(t) + \phi(s)\cdot\left((-1)^k\psi(t) + \psi(B')\right).$$

(keeping note not to use commutativity, or assume it).

$$\dots = \phi\left(\partial\sigma|_{[v_0,\dots,v_{k+1}]}\right)\cdot\psi(t) + (-1)^k\phi(s)\cdot\psi\left(\partial\sigma|_{[v_k,\dots,v_N]}\right)$$

$$= (\delta\phi\smile\psi)\sigma + (-1)^k(\phi\smile\delta\psi)\sigma$$

$$= \delta\phi\smile\psi + (-1)^k(\phi\smile\delta\psi))\sigma. \qquad \square.$$

- <u>Corollary A</u>: $Z^k \times Z^l \xrightarrow{\smile} Z^{k+l}$.
- <u>Corollary B</u>: $B^k \times Z^l \xrightarrow{\smile} B^{k+l}$, and $Z^k \times B^l \xrightarrow{\smile} B^{k+l}$.

- proof: Exercise.