Lecture 16 2022-02-14

Scribe

Coho w6 l1

①

Suppose that X is a topological space, R a commutative ring w/ 1_R.

Define $H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R)$

$$([\varphi],[\psi]) \longmapsto [\varphi \cup \psi]$$

this is the Cup Product on Cohomology.

<u>Lemma</u>

The Cup Product on H^* is

(i) well-def;

ii) R-bilinear;

iii) associative;

iv) $[\varepsilon]$ is a unit for $\cup$.

Proof – exercise (simple).

Lre

<u>Lemma 3.10</u>: Suppose $f: X \to Y$ is continuous. Then for $[\varphi], [\psi]$ in $H^k(Y)$, $H^l(Y)$ resp., we have

~~$[\varphi \cup \psi]$~~

$$f^*([\varphi] \cup [\psi]) = f^*([\varphi]) \cup f^*([\psi])$$

Proof

$$\begin{aligned} f^*([\varphi]\cup[\psi]) &= f^*([\varphi \cup \psi]) && \text{(def of } \cup) \\ &= [f^*(\varphi \cup \psi)] && \text{(def of } f^* \text{ in } H^*) \\ &= [(\varphi \cup \psi) f_*] && \text{(def of } f^* \text{ on } C^*) \end{aligned}$$

$= [(\varphi f_*) \cup (\psi f_*)]$ (def of $\cup$ on C^*)
$= [(f^*\varphi) \cup (f^*\psi)]$ (def of f^* on C^*)
$= [f^*\varphi] \cup [f^*\psi]$ (def of $\cup$ on H^*)
$= f^*[\varphi] \cup f^*[\psi]$ (def of f^* on H^*). $\square$

27 R-algebras

Suppose A is a ring (not neccessarily commutative), w/ 1_A, & an R-module. Then we call A an R-algebra.

Rmk The image of R in A under

$$R \longrightarrow A, \quad r \longmapsto r \cdot 1_A$$

is central.

$$(r 1_A) \cdot a = r 1_A \cdot a = a \cdot (r \cdot 1_A)$$

Example $\mathbb{Z}/2\mathbb{Z}$ is a $\mathbb{Z}$-algebra

Example polynomial rings,

$$R[x], \quad R[x,y].$$

Definition Suppose $X \neq \emptyset$.
Define $H^*(X;R) = \bigoplus_{k=1}^{\infty} H^k(X;R)$
We call $H^*(X;R)$ the Cohomology Ring w/ coeffs in R.

Rmk If $X = \emptyset$ then $H^* = 0$ so ~~the~~
$1 = 0$ ...

Lemma $H^*(X;R)$ is an R-algebra.

Also: $H^*(X;R)$ is given w/ a Graded Graded
Algebra Structure!

Examples:

i) $H^*(pt;R) \cong R$

ii) $H^k(S^1;R) \cong R \oplus R\cdot\omega$
$$\cong R[\omega]/\omega^2$$
where ω is the winding cocyle
(carefull what is ω when $R=\mathbb{Z}$?)

iii) $H^*(\mathbb{T}^2;R) \cong \underbrace{R}_{0\text{-dim}} \oplus \underbrace{R\alpha \oplus R\beta}_{1\text{-dim}} \oplus \underbrace{R(\alpha\cup\beta)}_{2\text{-dim}}$ 2-dim

where α, β are the winding classes, classes

(†) $\alpha\cup\beta = -\beta\cup\alpha$,
$\alpha\cup\alpha = 0 = \beta\cup\beta$.

28 Graded Commutativity

Thm 3.11 Suppose $\alpha\in H^k$, $\beta\in H^l$, then
$$\beta\cup\alpha = (-1)^{kl}(\alpha\cup\beta).$$

– Question: Is there an intuitive explanation for (†)?

Answer: For $\alpha\cup\beta = -\beta\cup\alpha$, we can think of moving the 1 part (the 1-simplex) the 1-simplex past β, which make 1 swap & so 1 multiplication by -1.

For $\alpha\cup\alpha=\beta\cup\beta=0$, this is not always true, so there isn't one.

Corollary of 3.11:

S'pose $\alpha \in H^k$. If k is even,

$$\alpha \cup \alpha = (-1)^{k^2} \alpha \cup \alpha = \alpha \cup \alpha.$$

If k is ~~x~~ odd,

$$\alpha \cup \alpha = (-1)^{k^2} \alpha \cup \alpha = -\alpha \cup \alpha$$

so $2 \cdot (\alpha \cup \alpha) = 0$,

& so either $2 = 0$ or $\alpha \cup \alpha = 0$ or $\alpha \cup \alpha$ is 2-torsion.

Answer to Question: In a polynomial ring, say $R[x]$, everything is generated by x.

Similar things are not in general true for other graded rings:

Example: Fix k, l w/ $k \neq l$, $k, l > 0$, $k < l$ say.

Then

$$H^*(S^k \times S^l; R) \cong H^*(S^k; R) \otimes H^*(S^l; R)$$
$$\cong R[x]/x^2 \otimes R[y]/y^2$$
$$\cong R \oplus Rx \oplus Ry \oplus R(x \cup y)$$

where $\deg(x) = k$, $\deg(y) = l$

(the degree of x, $\deg(x)$, is just the degree of the cohomology gp that contains it)

We will come back to this when we get to the <u>Künneth Formula</u>.

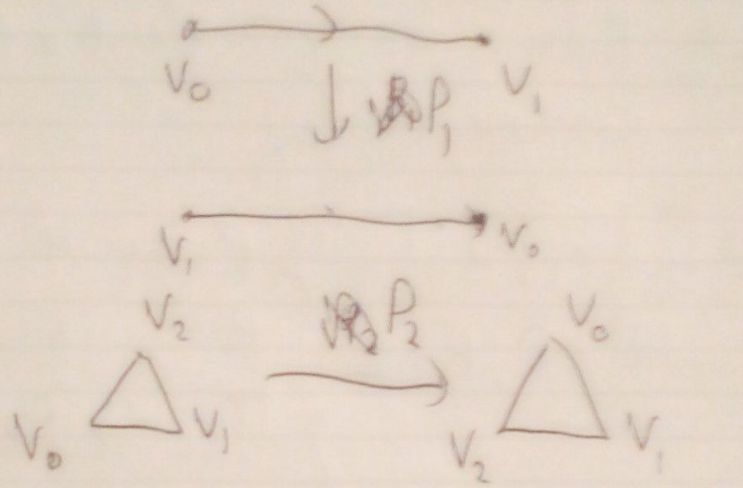
Sketch Proof of Thm. 3.11:

Given $\sigma : \Delta^k \to X$,

~~who~~ we have vertices $v_0, \dots, v_k$ for Δ^k.

Let $P_k : \Delta^k \to \Delta^k$ be the linear map map

~~that~~ that reverses the order of the v_i



k+1 \choose 2

Note that P is the product of $\binom{k+1}{2}$ reflections,

$\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \ \| \quad k+1+k+k-1+\dots+1.$

Define $\bar{\sigma} = \sigma \circ P$, $\varepsilon_k = (-1)^{\binom{k+1}{2}}$,

& $\rho_k(\sigma) = \varepsilon_k \bar{\sigma}$

~~(note ρ denotes a map on cha~~

Lemma ① $\rho_* : C_*^{sing} \to C_*^{sing}$ is chain map

Lemma ② ρ_* is chain homotopic to $\mathbb{1}$

So dualise to get ρ^*.

Now compute $\rho^*(\varphi \cup \psi)(\sigma)$ &

$\rho^*(\psi \cup \varphi)(\sigma)$,

& examine signs. $\square$

[Reading Exercise - See Hatcher]

29 Graphs

Define a graph Γ ~~be~~ to be a 1-dim (connected) CW complex. ~~$T \subseteq \Gamma$~~

A subcomplex $T \subseteq \Gamma$ is a Spanning Tree iff $\Gamma^{(0)} \subseteq T$ & T has no cycles.

Exercise This is equivalent to T being contractable.

Exercise [Suppose Γ is finite if you like]

$$H^k(\Gamma; R) \cong \begin{cases} R & \text{if } k=0 \\ \prod_{e \in \Gamma - T} R e^* & \text{if } k=1 \\ 0 & \text{o/w} \end{cases}$$

Check $e^* \cup f^* = 0$ for non-tree edges.