Lecture 17 2022-02-15

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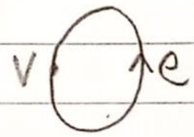
(29) Graphs and two complexes [spines and lens spaces]

We define $X_m = X$ to be the following two complex

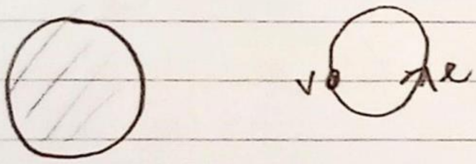
$X^{(0)}$ is a point v

$X^{(1)} \cong S^1$ is v and one one-cell e



$X_m = X = X^{(2)}$

and there is exactly one two-cell, f.



$F \cong D^2$

we must fix $d : \partial D^2 \cong S^1 \to (v \cup e) \cong S^1$

$$d : S^1 = \partial D^2 \to S^1 = X^{(1)}$$
$$z \mapsto z^m$$

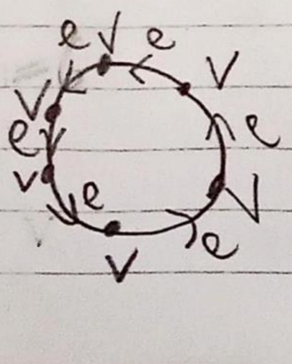
Examples: $m = 1$, $X \cong D^2$

$m = 0$, $X \cong S^1 \vee S^2$

$m = 2$, $X \cong \mathbb{RP}^2$

$m = 3$, X is the spine of $L(3,1)$ the lens space

Picture $m = 6$



Fix R and obtain cellular chain groups.

$$0 \xrightarrow{0} \underset{\langle f \rangle}{R} \xrightarrow{\times m} \underset{\langle e \rangle}{R} \xrightarrow{0} \underset{\langle v \rangle}{R} \xrightarrow{0} 0$$

Thus:

$$H_k^{cell}(X_m, R) \cong \begin{cases} R & k=0 \\ R/mR & k=1 \\ Ann_R(m) & k=2 \\ 0 & \text{else} \end{cases}$$

Now dualise

$$0 \xleftarrow{0} \underset{\langle f^* \rangle}{R} \xleftarrow{m} \underset{\langle e^* \rangle}{R} \xleftarrow{0} \underset{\langle v^* \rangle}{R} \xleftarrow{0} 0$$

Thus:

$$H^k_{cell}(X_m, R) = \begin{cases} R & k=0 \\ Ann_R(m) & k=1 \\ R/mR & k=2 \\ 0 & \text{else} \end{cases}.$$

If $\frac{R}{mR} \cong 0$ then the cup product is trivial.
If $Ann_R(M) = 0$ then likewise.

Example: Take $R = \frac{\mathbb{Z}}{m\mathbb{Z}}$. Here $H^k_{cell}(X_m, \frac{\mathbb{Z}}{m\mathbb{Z}})$

$$\cong \begin{cases} \mathbb{Z}/m\mathbb{Z} & k=0,1,2 \\ 0 & \text{else} \end{cases}.$$

So we must compute $[e^n] \smile [e^n]$. This must be some multiple of $[f^n] \in H^2$.

Unfortunately we do not have a definition of the cellular cup product. So we the isomorphism $H^n_{cell} \cong H^n_\Delta$