COHOMOLOGY AND POINCARÉ DUALITY LECTURE 18 2022 02 17

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EXERCISE:

(1) compute CW structures on $\mathbb{RP}^n$, $\mathbb{CP}^n$, $n \le \infty$.

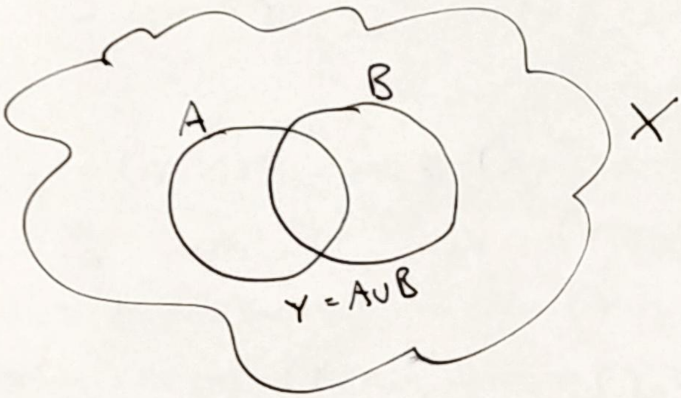
(2) Use these to compute H_* and H^* (the groups not the cup product structure).

30. RELATIVE CUP PRODUCTS

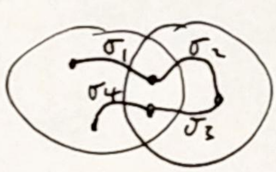
Suppose (X, Y) is a pair and $\mathcal{V} = \{A, B\}$ is an excisive cover of Y (that is, $A, B \subset Y$ and $Y \subset \operatorname{int}(A) \cup \operatorname{int}(B)$).

OR, X is CW and $Y = A \cup B$ are all subcomplexes.

Picture



Define: $C_k^{\mathcal{V}}(Y) = \{k\text{-chains in } Y \text{ subordinate to } \mathcal{V}\}$



Claim/Defn: $0 \to C_*^{\mathcal{V}}(Y) \xrightarrow{i_*} C_*(X) \xrightarrow{q_*} C_*^{\mathcal{V}}(X, Y) \to 0$

is short, exact, and split.

We dualise to obtain

$$0 \leftarrow C^*_{\mathcal{V}}(Y) \leftarrow C^*(X) \leftarrow C^*_{\mathcal{V}}(X, Y) \leftarrow 0$$

Lemma: $H^*(X,Y;Q) \cong H^*_\vee(X,Y;Q)$ is a natural isomorphism.

Proof: Long exact sequences and the five lemma. $\square$ (EXERCISE).

Hatcher says: cochains that "vanish" are better than cochains on relative chains.

What on Earth does this mean? This is what it means:

Define: $D^k(X,Y) := \{\varphi \in C^k(X) \mid \varphi(\sigma) = 0 \text{ if } \sigma \in C_k(Y)\}$ (take $Q = R$). These are the cochains that vanish.

Recall ~~that~~

$$0 \to C_k(Y) \xrightarrow{i_k} C_k(X) \xrightarrow{q_k} C_k(X,Y) \to 0$$

dualise to obtain

$$0 \leftarrow C^k(Y) \xleftarrow{i^k} C^k(X) \xleftarrow{q^k} C^k(X,Y) \leftarrow 0$$

Note that $D^k(X,Y) = \ker(i^k)$. Also, $D^k(X,Y) \xleftarrow{q^k} C^k(X,Y)$ is an isomorphism.

Similarly: $D^k_\vee(X,Y) \xleftarrow{q^k} C^k_\vee(X,Y)$ for the subordinate case where
$D^k_\vee(X,Y) = \{\varphi \in C^k(X) \mid \varphi(\sigma) = 0 \text{ for } \sigma \in C_k(A) \text{ or } \sigma \in C_k(B)\}$.

Define: $\rho : D^k(X,Y) \hookrightarrow D^k_\vee(X,Y)$

$$\varphi \longmapsto \varphi$$

Lemma: ρ induces isomorphism on H^* (so $H^*_\vee(X,Y) \cong H^*(X,Y)$).

Proof: ~~of course~~ LES & 5-lemma (EXERCISE). $\square$

~~Theorem~~:

Theorem: The absolute (means "regular") cup product restricts to give the relative cup product (use $Q = R$ coefficients)

$$H^k(X,A) \times H^l(X,B) \longrightarrow H^{k+l}(X,Y)$$

Proof: Replace the groups $C^*(X,A)$, $C^*(X,B)$, $C^*(X,Y)$ by the D groups, and restrict to get

$$\begin{tikzcd} C^k(X) \times C^l(X) \arrow[r, "\smile"] & C^{k+l}(X) \\ D^k(X,A) \times D^l(X,B) \arrow[u, hook] \arrow[r, "\smile"] & D_v^{k+l}(X,Y) \arrow[u, hook] \end{tikzcd}$$

~~All vertical maps induce isomorphism on cohomology H^*.~~ $\square$

(WARNING: What is natural?)

This gives the relative cup product on ~~chain~~ "cochains that vanish" as the restriction of the absolute cup product. $\square$

31. TENSOR PRODUCTS

Suppose $M, N \in \underline{Mod}_R$.

Define: $F(M,N)$ to be the free R-module generated by the set $M \times N$.

Define: $S(M,N) \subset F(M,N)$ to be the submodule generated by

(i) $(m+m', n) - (m,n) - (m',n)$

(ii) $(m, n+n') - (m,n) - (m,n')$

(iii) $(rm, n) - r(m,n)$

(iv) $(m, rn) - r(m,n)$

Define: $M \otimes_R N = \dfrac{F(M,N)}{S(M,N)}$. We call the image of (m,n) in $M \otimes_R N$ a pure tensor and write $m \otimes n \in M \otimes_R N$.

Remark: Most elements of $M \otimes_R N$ need not be pure.

EXAMPLE: $R^2 \otimes R^2$. $\Rightarrow a \otimes c - b \otimes d$, is not a pure tensor.
$\langle a,b \rangle \quad \langle c,d \rangle$

Note: $(R)^k \oplus (R)^l \cong (R)^{k+l}$

$(R)^k \otimes (R)^l \cong (R)^{kl}$ (EXERCISE).

Remark: $R \otimes_R R \cong R$.

Suppose $\mathcal{A}, \mathcal{B}$ are graded R-algebras. (So if $a, a' \in \mathcal{A}$ of degrees k and l then $a'a = (-1)^{kl} aa'$.)
We define the graded ~~R-module~~ R-algebra
$\mathcal{A} \otimes_R \mathcal{B}$ as before with multiplication

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{\deg(b)\deg(a')} (aa') \otimes (bb')$$

for a, b, a', b' ~~pure~~ homogeneous of degrees $\deg(a), \ldots$.
Note: This gives a multiplication on pure tensors so extend linearly to all tensors.

Theorem (Künneth) Let $H^k(Y;R)$ be fin. gen. free $\forall k$. ~~Then~~ and X, Y CW complexes. Then

$$H^*(X \times Y; R) \cong H^*(X;R) \otimes_R H^*(Y;R).$$

~~Exam~~