Scribe Notes

21/07

Lecture 19

(31) Tensor Products

Example · $H^*(\mathbb{T}^2, \mathbb{Z}) = H^*(S^1 \times S^1, \mathbb{Z})$ is isomorphic to the graded tensor product of two copies of $\mathbb{Z}[\alpha]/\alpha^2$.

· That is, if α, β generate $H^1(\mathbb{T}^2, \mathbb{Z})$, then $\alpha \cup \beta = -\beta \cup \alpha$.

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(32) Cross Product: Spose X, Y are spaces.

We have projections:

$$P_X : X \times Y \to X$$

$$P_Y : X \times Y \to Y$$

Diagram:

$$\begin{tikzcd} & X \times Y \arrow[dl, "P_X"'] \arrow[dr, "P_Y"] & \\ X & & Y \end{tikzcd}$$

· In cohomology $(Q = R)$

$$\begin{tikzcd} & H^*(X \times Y; R) & \\ H^*(X;R) \arrow[ur, "p_X^*"] & & H^*(Y;R) \arrow[ul, "p_Y^*"'] \end{tikzcd}$$

· Define a bilinear map

$$H^k(X;R) \times H^l(\cancel{X}Y;R) \xrightarrow{\times} H^{k+l}(X \times Y; R)$$

$$(\varphi, \psi) \longmapsto p_X^*(\varphi) \cup p_Y^*(\psi)$$

· ~~Define~~

· Define (again called cross product)

$$\mu : H^*(X) \otimes H^*(Y) \xrightarrow{\bar{\times}} H^*(X \times Y)$$

to be the extension of $\times$ to the tensor product.

Exercise μ is a homomorphism of R-algebras.

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(33) <u>Plan of proof of 3.15 (Kunneth formula)</u>

we will ~~[illegible]~~ leave this for a bit.

34 Prop 3.17 (promised in lecture 12)

~~props~~
Prop 3.17 Spose h, k are cohom theories

$$h, k : \underline{Pairs}_{cw} \longrightarrow \underline{Grad}^{R}$$

and $\mu : h \to k$ is a natural transformation

with $\cancel{\mu_{(pt,\emptyset)}}$ $\mu_{(pt,\emptyset)} : h(pt, \emptyset) \longrightarrow k(pt, \emptyset)$

is an isomorphism.

<u>Then</u>: $\mu_{(X,A)}$ is an isom for all

$(X, A) \in \underline{Pairs}_{cw}$

Pf ① Spose $\mu_{(X,\emptyset)}$ is an isomorphism for all X.

Then we have :

$$\begin{tikzcd}
h^n(A) \arrow[d, "\mu^n_A", "\overset{1}{\cong}"'] & h^n(X) \arrow[l] \arrow[d, "\mu^n_{X}", "\overset{1}{\cong}"'] & h^n(X,A) \arrow[l] \arrow[d, "\mu^n_{(X,A)}", "\overset{5s.}{\cong}"'] & h^{n-1}(A) \arrow[l] \arrow[d, "\mu^{n-1}_A", "\overset{1}{\cong}"'] & h^{n-1}(\overset{X}{\cancel{A}}) \arrow[l] \arrow[d, "\mu^{n-1}_X", "\overset{1}{\cong}"'] \\
k^n(A) & k^n(X) \arrow[l] & k^n(X,A) \arrow[l] & k^{n-1}(A) \arrow[l] & k^{n-1}(\underset{X}{\cancel{A}}) \arrow[l]
\end{tikzcd}$$

· Since h, k are cohom thm the LES axiom gives exact rows. Since μ is natural we get commuting squares.

Assumption ① implies μ_X^*, μ_A^* are Isom.

· So 5-lemma implies $\mu_{(X,A)}^*$ are Isom.

· Thus it suffices to show that μ_X is an isom for all $X \in \underline{TOP}_{cw}$.

② Suppose X is finite dimensional.

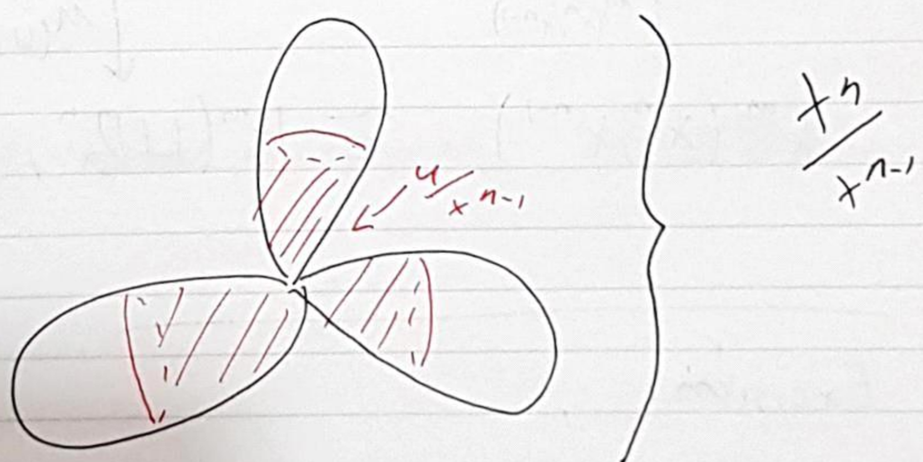
Induct on $n = \dim X$.

Base case: $\dim(X) = 0$, so X is a disjoint union of points. By hypothesis $h(pt) \underset{\mu}{\cong} k(pt)$ and by the disjointness axiom of cohomology we are done.

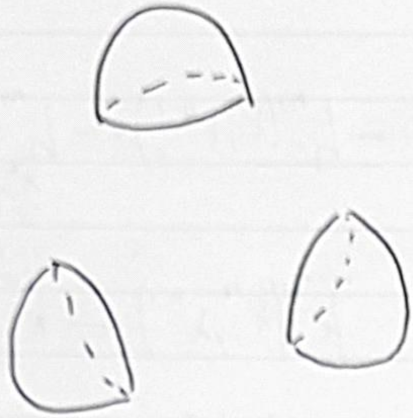
Inductive Step:

$$\begin{tikzcd}
h^m(X^{n-1}) \arrow[d, "\cong" ', "\mu^m_{X^{n-1}} \, (2)"] & h^m(X^n) \arrow[l] & h^m(X^n, X^{n-1}) \arrow[l] & h^{m-1}(X^{n-1}) \arrow[l] \arrow[d, "\cong" ', "\mu^{m-1}_{X^{n-1}} \, (2)"] & h^{m-1}(X^n) \arrow[l] \\
k^m(X^{n-1}) & k^m(X^n) & k^m(X^n, X^{n-1}) & k^{m-1}(X^{n-1}) \arrow[l] & k^{m-1}(X^n) \arrow[l]
\end{tikzcd}$$

By 5 lemma, ~~we~~ $\mu^*_{X^n}$ is an isom iff $\mu^*_{(X^n, X^{n-1})}$. Note, (X^n, X^{n-1}) is a good pair, so ~~we~~ there is a nbhd $U \subset X^n$ that deformation retracts to X^{n-1}:



· We ~~Now excise~~ now excise X^{n-1}/X^{n-1} from X^n/X^{n-1} using U to get



By excision, $\cancel{h^m(X^n, X^{n-1}) \cong h^m(\sqcup D^n_\alpha, \sqcup \partial D^n_\alpha)}$

$\cancel{k^m(X^n, X^{n-1}) \cong k^m(}$

excision, homotopy invariance

$$\begin{array}{ccc} h^m(X^n, X^{n-1}) & \cong & h^m(\sqcup D^n_\alpha, \sqcup \partial D^n_\alpha) \\ \Big\downarrow \mu^m_{(X^n, X^{n-1})} & & \Big\downarrow \mu^m_{(\sqcup, \sqcup\partial)} \\ k^m(X^n, X^{n-1}) & \cong & k^m(\sqcup D^n_\alpha, \sqcup \partial D^n_\alpha) \end{array}$$

Excision:

$$h(X, A) \cong h(B, B \cap A)$$

with $A = \overline{U}$, $B = X^n - U$.

Apply disjointness, so it suffices to show

$\mu(D^n, \partial D^n)$ is an isomorphism.

<u>Now</u> Apply LES <u>again</u> to the pair $(D^n, \partial D^n)$

and apply homotopy $\mu_{D^n} \cong \mu_{pt}$ and

induction hyp, $\mu_{\partial D}$ is an isom:

$$\begin{array}{ccccccccc} h^m(\partial D^n) & \leftarrow & h^m(D^n) & \leftarrow & h^m(D^n, \partial D^n) & \leftarrow & h^{m-1}(\partial D^n) & \leftarrow & h^{m-1}(D^n) \\ \cong\downarrow \text{induction} & & \cong\downarrow \text{homotopy} & & \cong\downarrow \text{five} & & \cong\downarrow \text{induction} & & \cong\downarrow \text{homotopy} \\ k^m(\partial D^n) & \leftarrow & k^m(D^n) & \leftarrow & k^m(D^n, \partial D^n) & \leftarrow & k^{m-1}(\partial D^n) & \leftarrow & k^{m-1}(D^n) \end{array}$$

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<u>Last</u>: Deal with case $\dim(X) = \infty$.