Question at start: Suppose $\phi \in H^k$, $\psi \in H^l$ and $H^{k+l} \neq 0$. Suppose $H_{k+l} = 0$

How to decide if $\phi \smile \psi$ is trivial?

Answer: If $\phi = 0$ or $\psi = 0$, there is no issue.

Remark: This can happen when the Ext group is non-trivial by the Universal Coefficient Theorem.

Suggestion: Consider both $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ coefficients for $X = K^2$ the Klein Bottle.

$$\begin{tikzcd}
h^m(\partial D^n) \arrow[d, "\cong", "\text{induction}"'] & h^m(D^n) \arrow[l] \arrow[d, "\cong", "\text{homotopy}"'] & h^m(D^n, \partial D^n) \arrow[l] \arrow[d, "\cong", "\text{5-lemma}"'] & h^{m-1}(\partial D^n) \arrow[l] \arrow[d, "\cong", "\text{induction}"'] & h^{m-1}(D^n) \arrow[l] \arrow[d, "\cong", "\text{homotopy}"'] \\
k^m(\partial D^n) & k^m(D^n) \arrow[l] & k^m(D^n, \partial D^n) \arrow[l] & k^{m-1}(\partial D^n) \arrow[l] & k^{m-1}(D^n) \arrow[l]
\end{tikzcd}$$

$$\Rightarrow h^m(D^n, \partial D^n) \cong k^m(D^n, \partial D^n) \qquad \square$$

Now, deal with $\dim(X) = \infty$

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(34) <u>Proof of 3.17:</u>

(3) Dealing with $\dim(X) = \infty$

<u>Proof 1)</u> Look ~~at~~ under the hood at definitions of ~~the~~ h, k.

<u>This is cheating!</u>

<u>Proof 2)</u> Dualise the proof of 2.34 in Hatcher

<u>This is too long!</u>

roof 3) Exercise: Prove MV for h, k directly from the axioms for cohomology theories.

~~$h^n(A \cap B) \leftarrow h^n(A) \leftarrow$~~

$$\cdots \leftarrow h^n(A \cap B) \leftarrow h^n(A) \oplus h^n(B) \leftarrow h^n(A \cup B) \leftarrow \cdots$$

[A, B subcomplexes of X, $A \cup B = X$]

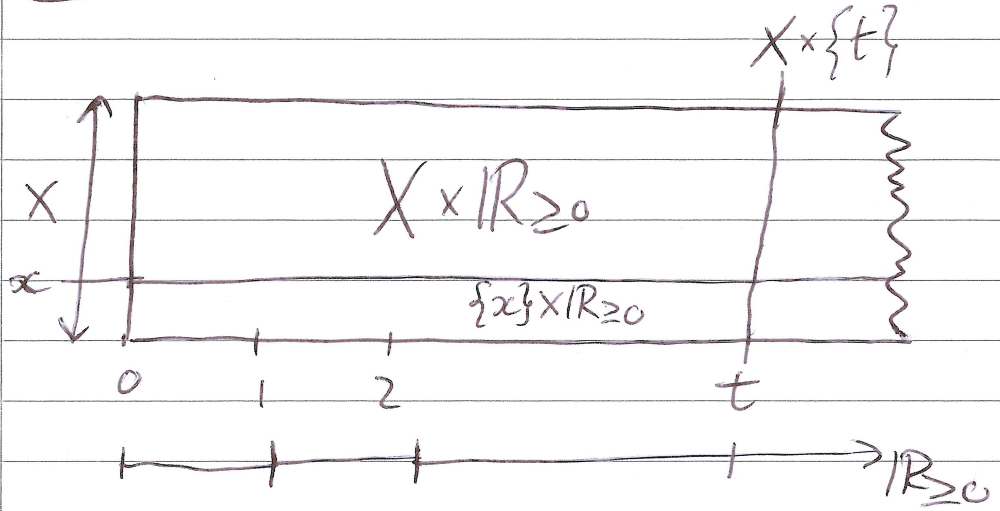
Hint, Only need LES and Excision and Elbow Grease

[cf Eilenberg, Mazur] Swindle

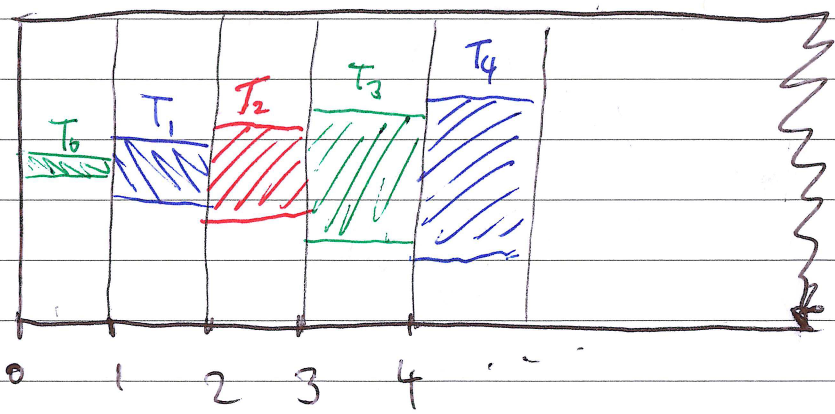
Suppose X is infinite dimensional and CW

Set $\mathbb{R}_{\geq 0} = [0, \infty]$

Picture:



$T_n = X^n \times [n, n+1]$



Define $T = \bigcup_{n \in \mathbb{N}} T_n$ the mapping telescope for X a CW-complex.

Define ~~$T_{even} = T_e = \bigsqcup_{n \text{ even}} T_n$~~

$$T_{even} = T_e = \bigsqcup_{n \text{ even}} T_n$$

$$T_{odd} = T_o = \bigsqcup_{n \text{ odd}} T_n$$

Note, $T = T_e \cup T_o$

Also, $T_e \cap T_o = \cancel{\bigsqcup_{n \in \mathbb{N}} X^{n-1} \times \{n\}}$

$$= \bigsqcup_{n \in \mathbb{N}} X^n \times \{n+1\}$$

<u>Difficult Exercise</u>: $T \simeq X \times \mathbb{R}_{\geq 0} \simeq X$

pair of homotopy equivalences.

Lemma: Direct product of isomorphisms is an isomorphism.

Giant Diagram: (Note $T_e \simeq \bigsqcup_{n \text{ even}} X^n$, $T_o \simeq \bigsqcup_{n \text{ odd}} X^n$)

$$
\begin{tikzcd}
h^n(T_o \cap T_e) \arrow[d, "\mu_n", "\cong"'] & h^n(T_o) \oplus h^n(T_e) \arrow[l] \arrow[d, "\mu_{T_o} \oplus \mu_{T_e} = \mu_o \oplus \mu_e", "\cong"'] & h^n(T) \arrow[l] \arrow[d, "\cong"', "\text{homotopy}"] & h^{n-1}(T_o \cap T_e) \arrow[l] \arrow[ddd, "\cong"] & h^{n-1}(T_o) \oplus h^{n-1}(T_e) \arrow[l] \arrow[ddd, "\cong"] \\
 & & h^n(X) \arrow[d, "\mu_X"] & & \\
 & & k^n(X) \arrow[d, "\cong"] & & \\
k^n(T_o \cap T_e) & k^n(T_o) \oplus k^n(T_e) \arrow[l] & k^n(T) \arrow[l] & k^{n-1}(T_o \cap T_e) \arrow[l] & k^{n-1}(T_o) \oplus k^{n-1}(T_e) \arrow[l]
\end{tikzcd}
$$

Here, the rows come from Mayer Vietoris.
The columns come from our previous proof that $\mu_X{}^n$ is an isomorphism.
Then μ_X is an isomorphism by the 5-lemma.

[3.15) Suppose X, Y are CW [and $H^l(Y)$ is f.g free for all l] Then

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$$

as graded R-algebras.

Plan: 1) Realise LHS and RHS as functors ~~to~~ h, k

2) Realise homomorphism as natural transformation

3) h, k are cohomology theories

4) $\mu : h \to k$ is an isomorphism on $(X, A) = (pt, \emptyset)$

5) Apply 3.17.

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1) Define h, k: Fix Y as in statement

$k^n(X, A) = H^n(X \times Y, A \times Y)$ [$Q = R$ everywhere]

$$h^n(X, A) = \bigoplus_{i+j=n} \left(H^i(X, A) \otimes H^j(Y) \right)$$

<u>Note</u>:

$$\begin{tikzcd} & (X \times Y, A \times Y) \arrow[dl, "p_{(X,A)}"'] \arrow[dr, "p_Y"] & \\ (X, A) & & Y \end{tikzcd}$$

Define $\mu : H^i(X,A) \otimes H^j(Y) \to H^{i+j}(X \times Y, A \times Y)$

by $(\phi, \psi) \mapsto p^*_{X,A}(\phi) \smile p^*_Y(\psi)$

Define μ to be the relative cross product.

So, $\mu : h \to k$ is a natural transformation.

Left: Verify h, k cohomology theories and μ_{pt} is an isom.