Cohomology 24th February 2022

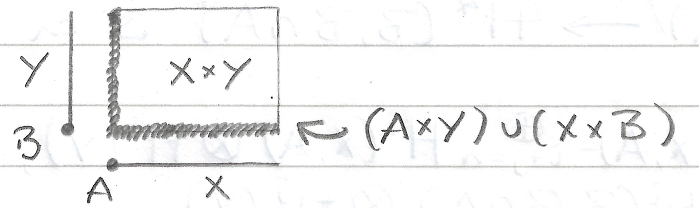
We are proving 3.15:
Let X, Y be CW complexes, $A \subseteq X$ a subcomplex, with $H^j(Y)$ finitely generated and free for all j.
Then the cross product $\mu: H^*(X,A) \otimes H^*(Y) \to H^*(X \times Y, A \times Y)$ is an isomorphism.

Theorem 3.18: (Fully relative version)
Let $(X,A), (Y,B) \in \text{Pairs}_{CW}$ with $H^j(Y,B)$ finitely generated and free for all j.
Then $\mu: H^*(X,A) \otimes H^*(Y,B)$

$$\downarrow$$

$$H^*(X \times Y, (A \times Y) \cup (X \times B))$$

is an isomorphism. (Proof in Hatcher)



Proof of 3.15:
① Define h and k

$$h^n(X,A) = \bigoplus_{i+j=n} H^i(X,A) \otimes H^j(Y)$$

$$k^n(X,A) = H^n(X \times Y, A \times Y)$$

$\mu: h \to k$ defined last time

② Show h and k are cohomology theories
③ Show $\mu_{(pt,\emptyset)}$ is an isomorphism
Exercise: Show ③ (Easy from definitions)

Now to show ②

① **Homotopy**: h and k are defined in terms of H^*, which satisfies homotopy

e.g. Suppose $f : (X,A) \xrightarrow{\simeq} (Z,C)$ is a homotopy equivalence.

Then $f \times Id_Y : (X\times Y, A\times Y) \xrightarrow{\simeq} (Z\times Y, C\times Y)$ is also a homotopy equivalence.

$f^* : H^*(Z,C) \to H^*(X,A)$ and

$(f \times Id_Y)^* : H^*(Z\times Y, C\times Y) \to H^*(X\times Y, A\times Y)$

are isomorphisms.

② **Excision**: Suppose $A, B \subseteq X$ are subcomplexes with $X = A \cup B$.

Then $i^* : H^*(X,A) \to H^*(B, B\cap A)$ is an isomorphism.

We have $h^n(X,A) = \bigoplus_{i+j=n} H^i(X,A) \otimes H^j(Y)$,

$h^n(B,B\cap A) = \bigoplus_{i+j=n} H^i(B,B\cap A) \otimes H^j(Y)$

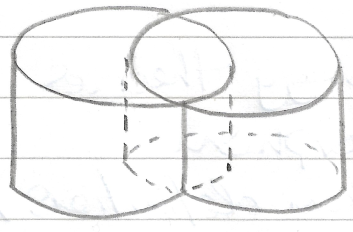
$i^* \otimes Id_Y^* : H^i(X,A) \otimes H^j(Y) \to H^i(B,B\cap A) \otimes H^j(Y)$

is an isomorphism, so this induces an isomorphism $i^* : h^n(X,A) \to h^n(B, B\cap A)$.

Note: $X\times Y = (A\cup B)\times Y = (A\times Y) \cup (B\times Y)$,

and $A\times Y, B\times Y \subseteq X\times Y$ are subcomplexes.

$(A\times Y)\cap(B\times Y) = (A\cap B)\times Y$

A  B

So since $k^n(X,A) = H^n(X\times Y, A\times Y)$

and $k^n(B, B\cap A) = H^n(B\times Y, (B\cap A)\times Y)$

$i^* : k^n(X,A) \to k^n(B,B\cap A)$

is an isomorphism.

III Long Exact Sequences

We want

$$\cdots \leftarrow h^{n+1}(X,A) \leftarrow h^n(A) \leftarrow h^n(X) \leftarrow h^n(X,A) \leftarrow h^{n-1}(A) \leftarrow \cdots$$

For k this holds because k^n is the homology of a pair of spaces $(X \times Y, A \times Y)$

Define: $\mathcal{L}^*$ to be the long exact sequence for (X,A)

So $\mathcal{L}^{3p+\varepsilon} = \begin{cases} H^p(A) & \varepsilon = 2 \\ H^p(X) & \varepsilon = 1 \\ H^p(X,A) & \varepsilon = 0 \end{cases}$

Define: $\mathcal{L}^*_n := \mathcal{L}^* \otimes_R H^n(Y) \cong \bigoplus_{\rho(n)} \mathcal{L}^*$

where $\rho(n) := \mathrm{rank}_R(H^n(Y))$

(The differentials are tensored with the identity)

Claim: $\mathcal{L}^*_n$ is a long exact sequence

Proof: $H^n(Y)$ is finitely generated and free

Suppose $\mathcal{M}^*$ is a long exact sequence.

Define $\mathcal{M}^*[n] := \mathcal{M}^{*-n}$ to be $\mathcal{M}^*$ shifted upwards by n. (i.e. $(\mathcal{M}^*[n])^p = \mathcal{M}^{p-n}$)

Define: $\mathbb{L}^* := \bigoplus_{n=0}^{\infty} \mathcal{L}^*_n[3n]$

This is again a long exact sequence.

$\mathbb{L}^k = \bigoplus_{n=0}^{\infty} \mathcal{L}^k_n[3n] = \bigoplus_{n=0}^{\infty} \mathcal{L}^{k-3n} \otimes H^n(Y)$

For Example:

$\mathbb{L}^0 = H^0(X,A) \otimes H^0(Y)$

$\mathbb{L}^1 = H^0(X) \otimes H^0(Y)$

$\mathbb{L}^2 = H^0(A) \otimes H^0(Y)$

$\mathbb{L}^3 = (H^1(X,A) \otimes H^0(Y)) \oplus (H^0(X,A) \otimes H^1(Y))$

This gives the desired LES

IV Disjoint Unions

This holds for k because k is the cohomology of a space.

This holds for h because $H^n(Y)$ is finitely generated and free.

Want to show that:

$$h^n\left(\coprod_\alpha (X_\alpha, A_\alpha)\right) \cong \prod_\alpha \left(h^n(X_\alpha, A_\alpha)\right)$$

Use the five lemma to reduce to the case $A_\alpha = \emptyset \ \forall \alpha$.

Use the fact:

$$\left(\prod_\alpha M_\alpha\right) \otimes \left(\prod_\beta N_\beta\right) \cong \prod_{\alpha,\beta} (M_\alpha \otimes N_\beta)$$

if the $\prod_\beta N_\beta$ is a finite product of copies of R.

<u>Exercise</u>: Check details.