②

(4A) $\mathbb{L}^2 = \bigoplus_{n=0}^{\infty} \mathcal{L}^{2-2n} \otimes H^n(Y)$

$= (\mathcal{L}^2 \otimes H^0(Y)) \oplus (\mathcal{L}^0 \otimes H^1(Y))$

$= H^0(A) \otimes H^0(Y) \oplus H^0(X,A) \otimes H(Y)$

This is the desired LES.

(IV) Disjoint Unions:

For k, we ~~simply note~~ use that it is defined using H^* & so this works

~~For h, this~~

exc Prove (IV) for h.

- WTS $h^n(\bigsqcup_\alpha (X_\alpha A_\alpha)) \cong \prod_\alpha h^n(X_\alpha A_\alpha)$.
- Use 5 lemma to reduce to $A_\alpha = \emptyset\ \forall \alpha$.
- Use the fact that for finite free M_α, N_β, $(\prod_\alpha M_\alpha \otimes \prod_\beta N_\beta) \cong \prod_{\alpha\beta} (M_\alpha \otimes N_\beta)$

WEEK 8

Coho w8 l1

Lecture 22 2022-02-28

Th Fully Relative Künneth Formula

S'pose $(X,A), (Y,B) \in \underline{\text{Pairs}}_{cw}$, & $H^k(Y,B)$ is finite gen free $\forall k$.

Then

$$H^*(X,A) \otimes H^*(Y,B) \xrightarrow{\text{cross prod}, \mu} H^*(X \times Y, X \times B \cup A \times Y)$$

is ~~an~~ an iso.

(thm) (Note the cup & cross prods are natural & related via projections

$$\begin{tikzcd} & X\times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd}$$

& the diagonal map

$$\begin{tikzcd} & X \arrow[d, "\Delta : x \mapsto (x,x)"] & \\ & X\times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

if $X=Y$.

Specifically, $\varphi \times \psi = p_X(\varphi) \cup p_Y(\psi)$

Pf - ?

~~Def~~ Ex The Cube

$I^1 := I := [-1,1] \subseteq \mathbb{R}$, $I^0 := \{0\} \subseteq \mathbb{R}$,

$I^{n+1} := I^n \times I \quad \forall n \in \mathbb{Z}_{\geq 0}$.

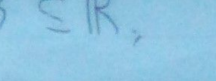
Note $I^i \times I^j \cong I^{i+j} \quad \forall i,j \in \mathbb{Z}_{\geq 0}$,

$\partial I^3 \cong S^2$ $\partial I \times I^2 \cup \partial I^2 \times I$ $\partial I^n \cong S^{n-1}$,

$I^n \cong D^n \subseteq \mathbb{R}^n$ (or $\mathbb{B}^n$ if you prefer)

$\partial I^{i+j} \cong (\partial I^i \times I^j) \cup (I^i \times \partial I^j) \quad \forall i,j.$

~~$H^i(I^i, \partial I^i) \otimes H^j(I^j, \partial I^j)$~~



Exercise

$S^3 \cong (S^1 \times \mathbb{B}^2) \cup (\mathbb{B}^2 \times S^1)$ = union of solid tori

Ex ctd. Suppose $i+j=n$

~~$H^i(I^i, \partial I^i) \otimes H^j(I^j, \partial I^j) \cong H^i(S^i) \otimes H^j(S^j)$~~

(Ex) Then we have

$$\begin{tikzcd}
H^i(I^i,\partial I^i)\otimes H^j(I^j,\partial I^j) \arrow[r, "\cong"] \arrow[d, "\mu"', "\cong"] & \tilde{H}^i(S^i)\otimes\tilde{H}^j(S^j) \arrow[dd, "\cong"] \\
H^n(I^n,(\partial I^i\times I^j)\cup(I^i\times\partial I^j)) \arrow[d, equal] & \\
H^n(I^n,\partial I^n) \arrow[r, "\cong"] & \tilde{H}^n(S^n)
\end{tikzcd}$$

Get a positive sign in the iso taking the pure tensor of generators, $\alpha_i\otimes\alpha_j$ to the generator α_n, where α_i is the right-handed generator of $\tilde{H}^i(S^i)$.

Ex **Projective Spaces**

Define $P^n=\mathbb{R}P^n$. This has 4 definitions:

~~(Def)~~ i) P^n is the <u>Space of Lines</u> in $\mathbb{R}^{n+1}$;

ii) $P^n=\dfrac{\mathbb{R}^{n+1}-\mathbb{R}^0}{\mathbb{R}^1-\mathbb{R}^0}$ (mod out by scaling);

iii) $P^n=S^n/_{x\sim -x}$ (mod out by the antipodal map);

iv) $P^n=B^n/(x\sim -x,\ x\in\partial B)$

<u>Exercise</u> (i)–(iv) are equivalent.

(Ex) In each def, the inclusion

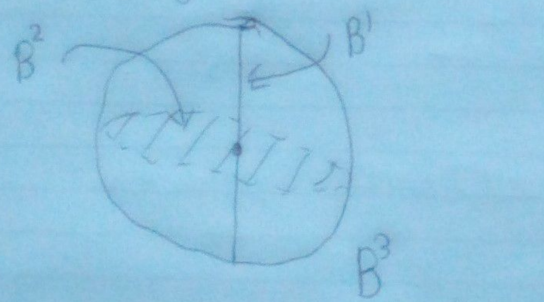
$$\mathbb{R}^{i+1} \hookrightarrow \mathbb{R}^{n+1}, \quad i<n \qquad (1^{st}\ i+1 \text{ coords})$$

gives an inclusion

$$P^i \hookrightarrow P^n, \quad \text{or} \quad P^i \subseteq P^n$$

(in (ii) $S^i \subseteq S^n \cap \mathbb{R}^{i+1}$,
& in (iv), $B^i \subseteq B^n \cap \mathbb{R}^{i+1}$)

Indeed, if $n=i+j$ and we use the inclusion $\mathbb{R}^{j+1} \hookrightarrow \mathbb{R}^{n+1}$ onto the <u>last</u> $j+1$ coords, we get $P^j \subset P^n$, & the subspaces $P^i, P^j \subseteq P^n$ defined in this way meet at a single pt:



$n=3, \ i=2, \ j=1.$

If $i<j$, we could also consider

$$P^i \subseteq P^j \subseteq P^n$$

So $\partial B^j \subseteq B^n$ gives an equatorial S^{j-1} in B^n & so gives (def (iv)) a P^{j-1} in P^n missing P^0

<u>Exercise</u> $P^n - P^{i-1} \simeq P^j$, where $n=i+j$.