Lecture 23: 01/03/2022

38. Projective Spaces

Note that Definition (4) of P^n shows that P^n has a CW structure with exactly one cell in each dimension.

Exercise: The attaching map $\varphi^k : \partial D^n \to P^{k-1}/P^{k-1}$ has degree $1 + (-1)^k$. [Hint: antipodal map is the composition of k reflections in $\mathbb{R}^n$.]

Thus, over $\mathbb{Z}$ we obtain

$$C_\bullet^{\mathrm{CW}}(P^n) : 0 \to \mathbb{Z} \to \dots \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0,$$

where the leftmost nonzero term is in degree n. With coefficients in $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ we have

$$C_\bullet^{\mathrm{CW}}(P^n; \mathbb{F}_2) : 0 \to \mathbb{F}_2 \xrightarrow{0} \dots \xrightarrow{0} \mathbb{F}_2 \to 0.$$

Thus

$$H_k^{\mathrm{CW}}(P^n; F_2) = \begin{cases} \mathbb{F}_2 & k = 0, 1, \dots, n \\ 0 & \text{else} \end{cases}.$$

Dualising, we get

$$H^k_{\mathrm{CW}}(P^n : \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & k = 0, 1, \dots, n \\ 0 & \text{else} \end{cases}.$$

Exercise: The inclusion $P^i \hookrightarrow P^n$ induces a surjection $\iota^* : H^*(P^n; \mathbb{F}_2) \twoheadrightarrow H^*(P^i; \mathbb{F}_2)$. [Hint: Use LES of pairs.]

Remark/Definition (5): Take $P^\infty = \bigcup_{n=0}^\infty P^n$.

Question: What is the CW structure on P^∞?

Answer: It is the structure obtained by declaring $(P^\infty)^{(k)} = P^k$.

Definition: Define

$$\mathbb{R}^\infty_{\mathrm{fin}} := \{x \in \mathbb{R}^\infty \mid \text{ all but finitely many entries of } x \text{ are } 0. \},$$

and

$$S^\infty_{\mathrm{fin}} := \{x \in \mathbb{R}^\infty_{\mathrm{fin}} \mid \|x\| = 1\},$$

also define

$$P^\infty := \frac{S^\infty_{\mathrm{fin}}}{x \sim -x}.$$

Exercise: S^∞_{fin} is contractible.

Question: Which norm are we using in the definition of S^∞_{fin}?

Answer: Since x has only finitely many nonzero terms, the L^2 norm is defined and we will use that. But it should not matter.

Theorem 3.19: We have

$$H^*(P^n; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1}),$$

with x in degree 1. Also,

$$H^*(P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x],$$

also with x in degree 1. A similar result for $\mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{P}^n_{\mathbb{H}}$ holds, though with $\mathbb{Z}$ coefficients and x in degree 2 (resp. 4).

Question: Why P^∞? What does it do for us?

Answer: (Bad reason) It is a relatively nontrivial but computatble example: we do not need spectral sequences.

(Better reason) It is a first example of what is called classifying spaces: i.e. if we are given a real manifold M, then homotopy classes of maps $M \to P^\infty$ parameterises real line bundles on M. Also the fact that its cohomology ring is a polynomial ring shows that there should be some polynomial invariants of line bundles. (Chern classes/characters?)

Reference: Characteristic Classes by Milnor.

Question: Why $\mathbb{R}^\infty_{\text{fin}}$ and not $\mathbb{R}^\infty_{\ell^2}$ (say)?

Answer: Eek!

Proof of 3.19: We induct on n. The base cases $n = 0, 1, 2$ have already been computed by hand. For the inductive step, suppose that P^{n-1} is done. Set α_{ii} to be the generator of $H^n(P^n)$ (the coefficient field will be suppressed from now on).

Since we have the surjection $\iota^* : H^*(P^n) \twoheadrightarrow H^*(P^{n-1})$, we can deduce that $\alpha_i \smile \alpha_j = \alpha_{i+j}$ whenever $i + j < n$. We are thus only left with the case $i + j = n$. For this, consider the giant diagram.

$$
\begin{array}{ccccc}
H^i(P_n) & \times & H^j(P^n) & \xrightarrow{\smile} & H^n(P^n) \\
\uparrow{\scriptstyle q^*} & & \uparrow{\scriptstyle q^*} & & \uparrow{\scriptstyle q^*} \\
H^i(P^n, P^n - P^j) & \times & H^j(P^n, P^n - P^i) & \xrightarrow{\smile} & H^n(P^n, P^n - P^0) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) & \times & H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) & \xrightarrow{\smile} & H^n(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^0) \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\
H^i(\mathbb{R}^i \times \mathbb{R}^j, (\mathbb{R}^i - \mathbb{R}^0) \times \mathbb{R}^j) & \times & H^j(\mathbb{R}^i \times \mathbb{R}^j, \mathbb{R}^i \times (\mathbb{R}^j - \mathbb{R}^0)) & \xrightarrow{\smile} & H^n(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^0) \\
{\scriptstyle i^*}\downarrow \; \uparrow{\scriptstyle p_i^*} & & {\scriptstyle i^*}\downarrow \; \uparrow{\scriptstyle p_j^*} & & \downarrow{\scriptstyle \mathrm{id}} \\
H^i(\mathbb{R}^i, \mathbb{R}^i - \mathbb{R}^0) & \times & H^j(\mathbb{R}^j, \mathbb{R}^j - \mathbb{R}^0) & \xrightarrow{\boxtimes} & H^n(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^0) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\
H^i(I^i, I^i - I^0) & \times & H^j(I^j, I^j - I^0) & \xrightarrow{\boxtimes} & H^n(I^n, I^n - I^0) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\
H^i(I^i, \partial I^i) & \times & H^j(I^j, \partial I^j) & \xrightarrow{\boxtimes} & H^n(I^n, \partial I^n)
\end{array}
$$

The second vertical arrow on each row is an isomorphism by excision. I have used $\boxtimes$ to indicate the relative cross product since $\times$ has been used too many times (and because its definition looks oddly reminiscent to the context in which this symbol is actually used). If we assume for the moment that the first vertical arrow in each row is an isomorphism, and that the entire diagram commutes, then chasing the images of (α_i, α_j) down to the last row we see that their relative cross product gives the generator for $H^n(I^n, \partial I^n)$, which would suffice to prove the theorem.

To finish the proof we would need to show that the q^* are isomorphisms, and that the diagram commutes. The only tricky bit of the second part has to do with the second row – all others are clear by 'naturality' of our constructions. The proof would most likely be finished the next lecture.