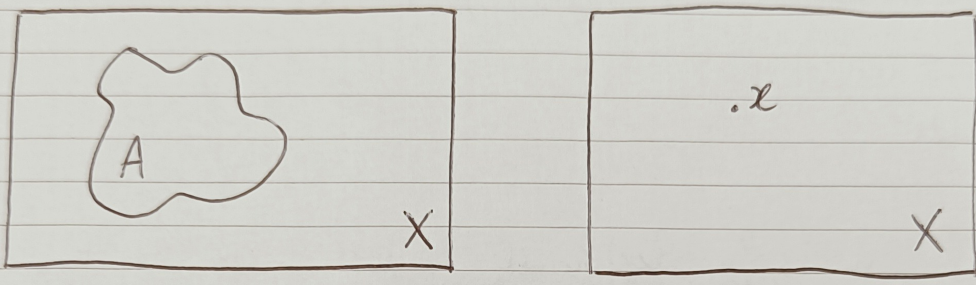
39 Local Homology and Manifolds

Definition. Suppose X is a topological space, fix $A \subset X$ (perhaps $A = \{x\}$). Define the local homology of X at A to be

$$H_*(X|A;R) := H_*(X, X-A;R)$$

We also write $H_*(X|x;R)$ if $A = \{x\}$.



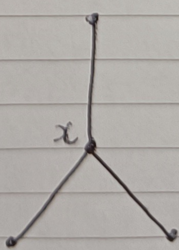
Example: $H_k(\mathbb{R}^n|x;R) = \begin{cases} R, & k=n \\ 0, & k\neq n \end{cases}$

Exercise. Prove this. Let B be a small closed ball containing x. Then $H_k(\mathbb{R}^n|x) = H_k(\mathbb{R}^n, \mathbb{R}^n - x)$ and by excision

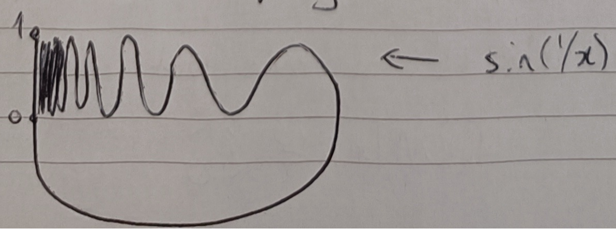
$$\begin{aligned} H_k(\mathbb{R}^n, \mathbb{R}^n - x) &\cong H_k(B^n, B^n - x) \\ &\cong H_k(B^n, \partial B^n) \qquad \text{(homotopy)} \\ &\cong \tilde{H}_k(B^n/\partial B^n) \\ &\cong \tilde{H}_k(S^n) \end{aligned}$$

Exercise. Let Y be the Y space, namely three intervals glued at x.

Compute $H_*(Y|x)$.

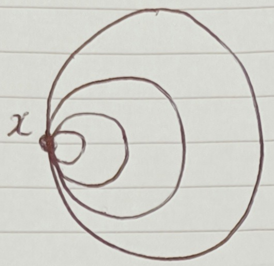


Exercise. Let C be the topologists circle.



Compute $H_*(C|x)$ where $x = (0, 1/2)$.

Exercise (hard). Let E be the earring space, that is for C_n the circle with centre $(\frac{1}{n}, 0)$ and radius $\frac{1}{n}$



$$E := \bigcup_{n \geq 2} C_n$$

Compute $H_*(E|x)$.

- Manifolds

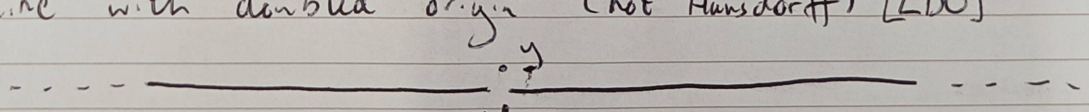
Suppose M is a topological space which has the following properties, for fixed n.

i) M is Hausdorff
ii) M is second countable
iii) $\forall x \in M$, $\exists U \subset M$ open, $x \in U$ and $U \cong \mathbb{R}^n$.

We call M an n-manifold.

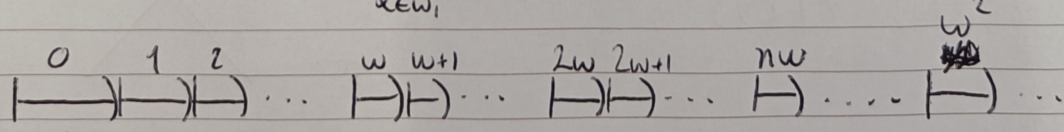
Examples: S^n, $\mathbb{R}^n$, $\mathbb{T}^n$, P^n, ...

Nonexamples: Y space, earring space, topologist's circle, the line with doubled origin (not Hausdorff) [LDO]



Compute $H_*(LDO|x)$ (Exercise)! Also the long ray

$$LR := \bigcup_{\alpha \in \omega_1} [0,1)$$



(not a manifold because it is not second countable!)

Exercise. If $x \in M^n$ a manifold then

$$H_k(M|x) \cong \begin{cases} R & k = n \\ 0 & k \neq n \end{cases}$$

Note this means the local ~~topology~~ homology of a manifold looks like the homology of $\mathbb{R}^n$, which makes sense because ~~the~~ manifolds are locally like $\mathbb{R}^n$.

40 Orientations

Suppose $A \subset B \subset X$, and then $\emptyset \subset X-B \subset X-A$. Let

$$\iota_{X,B,A} : (X, X-B) \to (X, X-A)$$

be the inclusion. This induces a homomorphism

$$\psi_{X,B,A} : H_*(X|B) \to H_*(X|A)$$

Special case: $\psi_{X,X,A} : H_*(X) \to H_*(X|A)$ is a homomorphism from homology of X to any local homology.

- The homology bundle

Fix M^n a manifold and R a commutative ring with 1, and with a topology on R.

Define M_R (the homology bundle)

$$M_R = \{ (x,r) \mid x \in M^n, r \in H_n(M|x) \}$$

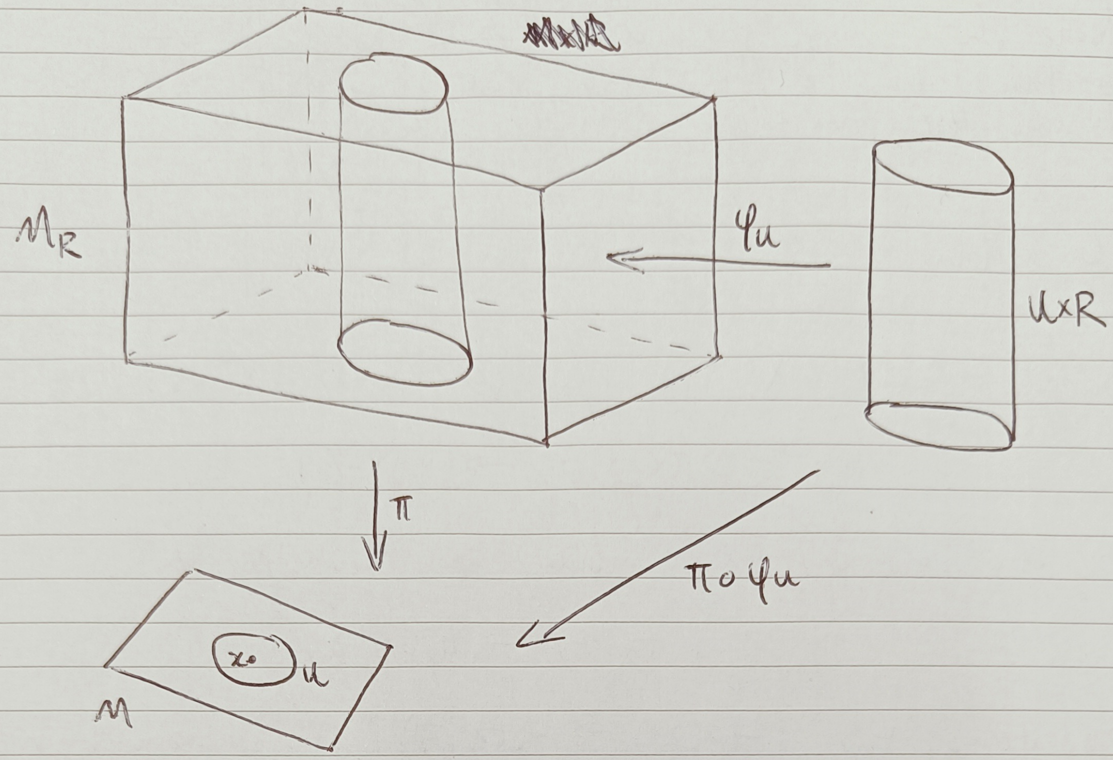
This is a fibre bundle with projection

$$\begin{array}{ccc} M_R & \xrightarrow{\pi} & M \\ \cup & & \\ (x,r) & \xmapsto{\pi} & x \end{array}$$

for a neighbourhood of x, $U \cong \mathbb{R}^n$ we require a homeomorphism

$$\varphi_U : U \times R \to \pi^{-1}(U)$$

so that $\pi \circ \varphi_U = \pi_U$ (projection on U).



A section $\sigma : M \to M_R$ of π is any continuous map so that $\pi \circ \sigma = Id_M$. [see Hatcher for construction of topology on M_R].

eg $\sigma_0 : M \to M_R : x \mapsto$

Call $r \in R$ a generator if $Rr = R$ (exactly the units since $1_R \in R$). Call $r \in H_n(M|x)$ a generator if it is the image of a generator under

$$H_n(M|x) \cong R.$$

Definition. Call $\sigma : M \to M_R$ an R-orientation if $\sigma(x)$ is a generator for all $x \in M$. If σ exists call M orientable.

Corollary: Every manifold is $\mathbb{Z}/2\mathbb{Z}$-orientable.

Proof. Exercise, see picture (double cover). $\square$