Cohomology 25.

Q.

Does H^*_{DeR} give a cohomology theory?
What is the definition of H^*_{DeR}

Lemma

Let M be a manifold, $R = \mathbb{Z}/2\mathbb{Z}$. Then M_R the cohomology bundle has exactly two sections (the zero section and the one section)

Corollary

Every manifold is $\mathbb{Z}/2\mathbb{Z}$ orientable.

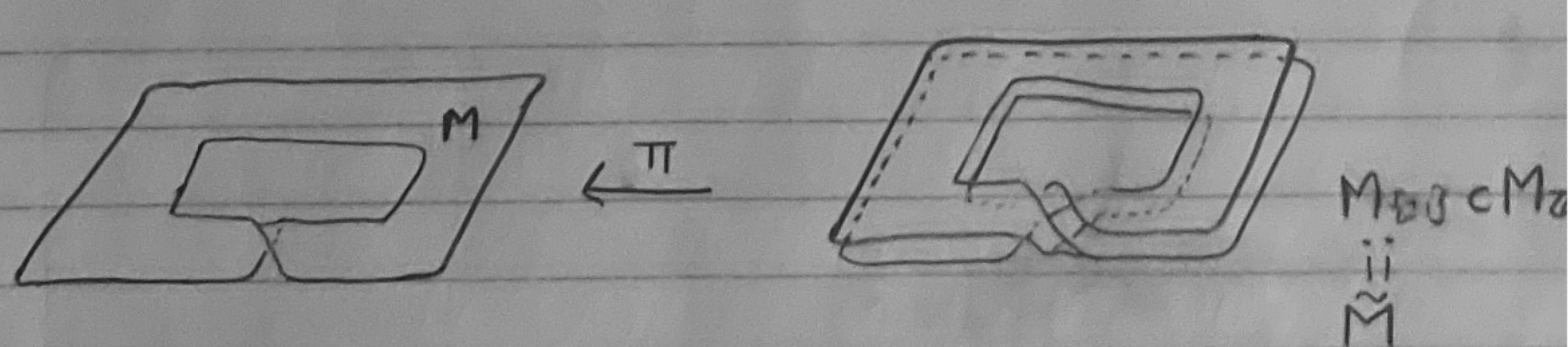
$R = \mathbb{Z}$ is more interesting

Examples

① $A = S^1 \times \mathbb{R}$ is $\mathbb{Z}$ orientable.
② M the Möbius band is not $\mathbb{Z}$ orientable.

Exercise

A surface S (two manifold) is $\mathbb{Z}$ orientable if and only if S' does not contain an embedded Möbius band.



Suppose M is connected Let $\tilde{M} = M_{\{\pm 1\}} \subset M_{\mathbb{Z}}$ be the components of $M_{\mathbb{Z}}$ containing the generators of $H_n(M|x)$

Proposition 3.25

M is $\mathbb{Z}$ orientable if and only if $\tilde{M}$ has two components.

Definition

$\tilde{M}$ is the orientation bundle over M

Note $\tilde{M}$ is a double cover of M hence:

Corollary

If $\pi_1(M)$ has no index two subgroup then M is $\mathbb{Z}$ orientable.

Proof

Exercise ■

Example

$S^3 \subset \mathbb{C}^2$, $S^3 := \{(z,w) : |z|^2 + |w|^2 = 1\}$

$L(3,1) = S^3 / (z,w) \sim \exp(\frac{2\pi i}{3}) \cdot (z,w)$

So $\pi_1(L(3,1)) \cong \mathbb{Z}/3\mathbb{Z}$

Proof of 3.25

$\mathbb{Z}$ orientable if and only if there is a non trivial section of M_R if and only if $\tilde{M}$ is the union of two such. ■

That is: M is $\mathbb{Z}$ orientable if and only if $M_{\mathbb{Z}} \cong M \times \mathbb{Z}$

$$\begin{array}{ccc} M_{\mathbb{Z}} & \cong & M \times \mathbb{Z} \\ \cup & & \cup \\ \tilde{M} & \cong & M \times \{\pm 1\} \end{array}$$

Q: What about other R (rings?)

"Often" $M_{\mathbb{Z}} \otimes_{\mathbb{Z}} R \cong M_R$.

Theorem 3.26

Suppose M^n is a closed (compact and without boundary), connected n-manifold.

(A) Suppose M is R orientable.
Then $\Psi_{M,x} : H_n(M|M) \to H_n(M|x)$ is an isomorphism for all x

(B) Suppose M is not R orientable
Then $\Psi_{M,x}$ is injective and
$im(\Psi_{M,x}) = \{r \in H_n(M|x) : r + r = 0\}$

(C) In any case $H_k(M) = 0$ for $k > n$

Recall

$H_n(M|A) = H_n(M, M-A)$

So $H_n(M|M) = H_n(M, \emptyset) \cong H_n(M)$

If $B \subset A \subset M$ then $M-A \subset M-B$ so
$\Psi_{A,B} : H_k(M|A) \to H_k(M|B)$ is the induced localisation homomorphism.

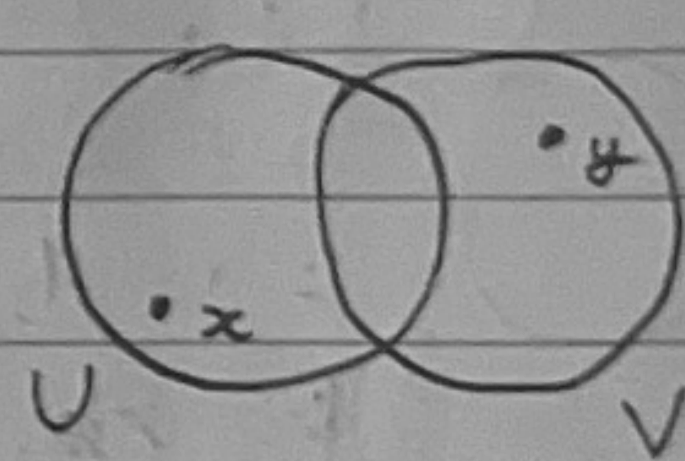
Lemma

Suppose M^n is connected and an n-manifold (without boundary). Suppose $A \subset M$ is compact

(A) Suppose $\sigma : M \to M_R$, $x \mapsto \sigma(x) = (x, \alpha_x)$ is a Section then there is a unique $\alpha_A \in H_n(M|A)$ So that $\Psi_{x,A}(\alpha_A) = \alpha_x$

(c) $H_k(M|A) = 0$ for $k > n$

This lemma is the "Patching" Lemma where we Patch together Local information (the α_x's) to get "bigger" data $\alpha_A \in H_n(M|A)$.



From α_x and α_y deduce α_U and α_V. Examine $\alpha_{U \cap V}$ to deduce (using relative Mayer Vietoris for H_*) existence and uniqueness of $\alpha_{U \cup V}$

Definition

A manifold M^n has a fundamental Class $[M] = \mu_M \in H_n(M;R)$ if M has $\Psi_{M,x}(\mu_M) = \mu_x \in H_n(M|x;R)$ is a generator for all $x \in M$

deR $\int_{[M]} \omega_{vol} = Vol(M)$

Corollary

M compact has fundamental class over R if and only if M is R-orientable.