LECTURE 29, TUE. 15 MAR 2022

4.5. DIRECT LIMITS

EXERCISES: compute the direct limits of

i. $\mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 0} \cdots$

ii. $\mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 1} \cdots$

iii. $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots$

iv. $\Big(\underset{G_m}{\mathbb{Z}} \xrightarrow{\times \frac{n}{m}} \underset{G_n}{\mathbb{Z}} \text{ if } m \mid n \Big)_{n \geq 0}$, e.g.

$$\begin{tikzcd}
 & \mathbb{Z}_6 & \\
\mathbb{Z}_2 \arrow[ur, "\times 3"] & & \mathbb{Z}_3 \arrow[ul, "\times 2"'] \\
 & \mathbb{Z}_1 \arrow[ul, "\times 2"] \arrow[ur, "\times 3"'] &
\end{tikzcd}$$

PROPOSITION (3.33): Suppose X is a space and R is a ring. Then

$$H^k_{cpt}(X;R) \cong \varinjlim_K H^k(X|K;R).$$

EXERCISE: Prove the above proposition.

REMARK: If $G = \varinjlim_\alpha G_\alpha$ then there are induced morphisms $\varphi_\alpha : G_\alpha \to G$.

46. LOCAL DUALITY MAPS

Suppose M^n is a connected n-manifold, and $\mu: M \to M_R$ is an R-orientation. Recall that 3.27 ~~gagra~~ gives a LOCAL FUNDAMENTAL CLASS $\mu_K \in H_n(M|K)$.

DEFINITION: Define the ~~DUALITY MAPS DO~~ LOCAL DUALITY MAPS

$$D_{M|K} : H^k(M|K) \longrightarrow H_{n-k}(M)$$
$$\varphi \longmapsto \mu_K \cap \varphi .$$

RECALL: The cap product can make local stuff global:

$$H_n(M|K) \times H^k(M|K) \xrightarrow{\cap} H_{n-k}(M).$$

Suppose $K \subset L \subset M$ with K, L compact. So $M-L \subset M-K$ and we have

$$H_n(M|L) \xrightarrow{\gamma_{L,K}} H_n(M|K).$$

DEFINITION: Define

$$\gamma^{K,L} : H^k(M|K) \longrightarrow H^k(M|L).$$

These are homomorphisms in the directed system

$$
\begin{tikzcd}
H^k_{cpt}(M) \arrow[rr, dashed, "D_M"] & & H_{n-k}(M) \\
\vdots & & \\
H^k(M|L) \arrow[rruu, "D_{M|L}"'] & & \\
H^k(M|K) \arrow[u, "\psi^{K,L}"'] \arrow[rruuu, "D_{M|K}"'] & &
\end{tikzcd}
$$

<u>DEFINITION</u>: Define

$$D_M := \varinjlim_{K} D_{M|K}.$$

WARNING: We need to define the direct limits of morphisms, but in the interest of brevity we are not doing this.

<u>LEMMA</u> (3.36, Mayer-Vietoris for H^*_{cpt}): Suppose M^n conn. n-mfd. Suppose $U, V \subset M$ are open and $M \subset U \cup V$. Then the following diagram (overleaf) has exact rows, and commutes (up to a sign $(-1)^k$).

$$
\begin{tikzcd}
\mathrm{A} \quad \cdots \arrow[r] & H_{n-k+1}(M) \arrow[r, "\partial"] \arrow[dr, phantom, "1\ (-1)^{k-1}"] & H_{n-k}(U \cap V) \arrow[r] \arrow[dr, phantom, "2"] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] \arrow[dr, phantom, "3"] & H_{n-k}(M) \arrow[r, "\partial"] \arrow[dr, phantom, "4\ (-1)^{k}"] & \cdots \\
\mathrm{B} \quad \cdots \arrow[r] & H^{k-1}_{cpt}(M) \arrow[r, "\delta"] \arrow[u, "D_M"] & H^{k}_{cpt}(U \cap V) \arrow[r] \arrow[u, "D_{U \cap V}"] & H^{k}_{cpt}(U) \oplus H^{k}_{cpt}(V) \arrow[r] \arrow[u, "D_U \oplus (-D_V)"] & H^{k}_{cpt}(M) \arrow[r, "\delta"] \arrow[u, "D_M"] & \cdots \\
\mathrm{C} \quad \cdots \arrow[r] & H^{k-1}(M|K \cup L) \arrow[r] \arrow[u] & H^{k}(U \cap V|K \cap L) \arrow[u] & H^{k}(U|K) \oplus H^{k}(U|L) \arrow[u] & H^{k}(M|K \cup L) \arrow[r] \arrow[u] & \cdots \\
\mathrm{D} \quad \cdots \arrow[r] & H^{k-1}(M|K \cup L) \arrow[r] \arrow[u, "\mathrm{id}"] & H^{k}(M|K \cap L) \arrow[r] \arrow[u, "\cong \text{ excision}"] & H^{k}(M|K) \oplus H^{k}(M|L) \arrow[r] \arrow[u, "\cong \text{ excision} \quad \cong \text{ excision}"] & H^{k}(M|K \cup L) \arrow[r] \arrow[u, "\mathrm{id}"] & \cdots
\end{tikzcd}
$$

Rows: A - D.

Commuting squares: 1 - 4.

PROOF: Fix $K \subset \mathcal{U}$, $L \subset V$ compct. Then Ⓒ is a collection of approximates to Ⓑ. Also by excision (or trivially) Ⓓ ≅ Ⓒ. Since the direct limit of commuting diagrams is also a commuting ~~diagrap~~ diagram, all that is left is to prove the commutativity of
① ⟶ Ⓐ.

Commutativity of ② and ③ hold at the chain/cochain level.

EXERCISE: Do the computation to show that ② and ③ commute at the (co)chain level.

Commutativity of ④ ~~fol~~ up to a sign of $(-1)^k$ follows from the Leibniz rule

$$\partial(\sigma \frown \varphi) = (-1)^k (\partial\sigma) \frown \varphi - \sigma \frown \delta\varphi. \qquad \square$$

<u>QUESTION</u>: In the definition of $\sigma_{k+l} \frown \varphi^k$ we took $\sigma \frown \varphi = \varphi(\sigma|[v_0, \dots, v_k])\, \sigma|[v_k, \dots, v_{k+l}]$.
Why not $= \varphi(\sigma|[v_l, \dots, v_{k+l}])\, \sigma|_{[v_0, \dots, v_l]}$
or some other middle face?
ANSWER: ~~This~~ ~~it~~ "just" (aha!) gives a change of sign. Consider precomposing <u>all</u> singular simplices by a permutation.

47. Poincaré Duality

We want to show that

$$D_M : H^k_{cpt}(M) \to H_{n-k}(M)$$

is an isomorphism. The proof is very similar to that of 3.27. That is, we induct on open sets decomposing M.

1. $M = U \cup V$. If PD holds for $U, V, U \cap V$ then use M-V and five-lemma (see previous diagram).
2. Same for finite unions.
3. Same for finite cover by charts.
4. If $K \subset M$ cpct then cover compact sets in charts.
5. Ascending unions: we will cover this.
6. Base case: $K \subset \mathbb{R}^n$ is convex ... so on.

$\boxed{5}$ is the step that is not similar to 3.27.

We can write $M = \bigcup_{i=0}^{\infty} U_i$, where U_i is a chart.

Now define $V_j = \bigcup_{i \leq j} U_i$. Thus V_j is a finite union (so ~~apply~~ apply (2)) and

$$M = \bigcup_{j=0}^{\infty} V_j$$

is an ascending union. Ascending union

play nicely with direct limits. That is,

$$D_M = \varinjlim_j D_{V_j} .$$

Remember that the actual definition is

$$D_M = \varinjlim_K D_{M|K}$$

More next time ~~time~~ in last lecture.