Please let me know if any of the problems are unclear, have typos, or have mistakes. Please turn in your solution to Exercise 1.6 on Friday (2020-01-21) before noon.

Exercise 1.1. Suppose that $X = S^2 \times S^4$ and $Y = \mathbb{CP}^3$.

1. Check that X and Y are compact, connected, oriented manifolds without boundary (of the same dimension).
2. Prove that $\pi_1(X)$ and $\pi_1(Y)$ are both trivial.
3. Give a CW–complex structure on each of X and Y.
4. Using this, or otherwise, compute the homology groups of X and of Y.
5. [Much harder.] Prove that X is not homeomorphic to Y.

Exercise 1.2. [Harder.] Repeat Exercise 1.1 with $X = S^2 \times S^2$ and $Y = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. Here $\#$ is the oriented connect sum operation, and $\overline{\mathbb{CP}}^2$ is $\mathbb{CP}^2$ with the opposite orientation.

Exercise 1.3. Suppose that G is a group. If x, y are elements of G then we define $[x, y] = xyx^{-1}y^{-1}$ to be their *commutator*. We define $[G, G] < G$ to be the subgroup generated by commutators.

- Prove that $[G, G]$ is a normal subgroup of G.

We define G^{ab} to be the quotient $G/[G, G]$. For $x \in G$ let $[x]$ be its image in G^{ab}.

- Prove that G^{ab} is abelian.

This gives a function $G \mapsto G^{\mathrm{ab}}$ from groups to abelian groups called *abelianisation*. If $f \colon G \to H$ is a homomorphism, we define $f^{\mathrm{ab}} \colon G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ via $f^{\mathrm{ab}}([g]) = [f(g)]$.

- Prove that f^{ab} is well-defined.
- Prove that abelianisation is a functor.

Exercise 1.4. Suppose that G is a group. We define the *centre* $Z(G) < G$ as follows.

$$Z(G) = \{x \in G \mid xy = yx \text{ for all } y \in G\}$$

Note that $Z(G)$ is abelian. Prove that passing to the centre is not functorial. That is, there is no function on morphisms making $G \mapsto Z(G)$ into a functor.

Exercise 1.5. Let **Pairs** be the category of pairs of topological spaces. Let $\mathbf{Mod}_{\mathbb{Z}}$ be the category of abelian groups. Fix k a positive integer. Define the functor $F \colon \mathbf{Pairs} \to \mathbf{Mod}_{\mathbb{Z}}$ by $F(X, A) = H_k(X, A)$. Define the functor $G \colon \mathbf{Pairs} \to \mathbf{Mod}_{\mathbb{Z}}$ by $G(X, A) = H_{k-1}(A)$. Prove that the connecting homomorphism $\delta \colon F \to G$ is a natural transformation.

Exercise 1.6. In each of the following, C_* is a chain complex of abelian groups. For each, decide if C_* is exact, compute the homology groups $H_*(C)$, and compute the cohomology groups $H^*(C;\mathbb{Z})$. If it is short exact, decide if it splits.

1. $0 \to \mathbb{Z} \to 0$
2. $0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z} \to 0$
3. $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z} \to 0$
4. $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \to 0$
5. $0 \to \mathbb{Z} \xrightarrow{u} \mathbb{Z}^2 \xrightarrow{v} \mathbb{Z} \to 0$ — where u is the column vector $\binom{p}{q}$, where v is the row vector $(q, -p)$, and where $\gcd(p, q) = 1$.

For the next two problems we fix an abelian group G and we define $A^* = \mathrm{Hom}(A, G)$.

Exercise 1.7. Suppose that $A \to B \to C \to 0$ is an exact sequence of abelian groups. Prove that $A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$ is also exact. [Thus we say that the functor $\mathrm{Hom}(\cdot, G)$ is *right exact*.]

Exercise 1.8. Suppose that $0 \to A \to B \to C \to 0$ is a split short exact sequence of abelian groups. Prove that $0 \leftarrow A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$ is again a split short exact sequence.