Please let me know if any of the problems are unclear, have typos, or have mistakes. Please turn in your solution to Exercise 2.4 on Moodle by noon Friday of week four (2022-02-04).

Exercise 2.1. Find a cell-structure on $\mathbb{RP}^n$. Using this, or otherwise, compute the homology groups of $\mathbb{RP}^n$.

Exercise 2.2. Find a cell-structure on $\mathbb{CP}^n$. Using this, or otherwise, compute the homology groups of $\mathbb{CP}^n$.

Exercise 2.3. Compute the homology groups of the following spaces.

1. $X = (S^n)^m$. [Some care is required when $n = 1$.]
2. $X = S^n \times S^m$.

Exercise 2.4. Suppose that $f\colon S^n \to S^n$ is a map. Review the definition of the *degree* of f from Section 2.2 (page 134) of Hatcher. Now prove that the degrees of the homomorphisms

$$f_n\colon H_n(S^n) \to H_n(S^n) \qquad \text{and} \qquad f^n\colon H^n(S^n;\mathbb{Z}) \to H^n(S^n;\mathbb{Z})$$

are equal.

Exercise 2.5. Suppose that X is a topological space. Suppose that $A, B \subset X$ are subsets so that X is contained in the union of the interiors of A and B. Suppose that Q is an abelian group. Let $C_*(A+B)$ be the chain complex consisting of all singular chains subordinate to the "cover" $\{A, B\}$. In the proof of excision we showed that the inclusion

$$i\colon C_*(A+B) \to C_*(X)$$

has a chain homotopy inverse ρ. Using this, or otherwise, prove that the dual homomorphism

$$i^\#\colon C^*(X;Q) \to C^*(A+B;Q)$$

has a chain homotopy inverse.

Exercise 2.6. With notation as in the previous problem. The Mayer-Vietoris long exact sequence, in homology, comes from the short exact sequence of chain complexes

$$0 \to C_*(A\cap B) \xrightarrow{\Delta} C_*(A) \oplus C_*(B) \xrightarrow{m} C_*(A+B) \to 0$$

Here the homomorphisms are given by $\Delta(c) = (c, c)$ and $m(c, d) = c - d$. Prove that the dual sequence

$$0 \leftarrow C^*(A\cap B;Q) \xleftarrow{\Delta^\#} C^*(A;Q) \oplus C^*(B;Q) \xleftarrow{m^\#} C^*(A+B;Q) \leftarrow 0$$

is exact.