Please let me know if any of the problems are unclear, have typos, or have mistakes. Please turn in your solution to Exercise 3.4 on Friday (2022-02-18) by noon, on Moodle. Below, if coefficients are not given, they are assumed to be $\mathbb{Z}$.

Exercise 3.1. Suppose that F is a vector space, over $\mathbb{R}$. Suppose that $\pi\colon E \to B$ is a *vector bundle* with fibre F. Compute the (co)homology groups of E in terms of those of B.

Exercise 3.2. [Challenge] Here is a hands-on definition of $\mathrm{UT}(S^n)$, the unit tangent bundle to the n–sphere.

$$\mathrm{UT}(S^n) = \{(u, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : |u| = |v| = 1, \langle u, v\rangle = 0\}$$

Here $\langle \cdot, \cdot\rangle$ is the usual inner product on $\mathbb{R}^{n+1}$. Compute the homology groups $H_*(\mathrm{UT}(S^n))$.

Exercise 3.3. A natural transformation $\delta\colon F \to G$ is called a *natural isomorphism* if there is another natural transformation $\epsilon\colon G \to F$ so that both $\delta \circ \epsilon$ and $\epsilon \circ \delta$ are identities. Now suppose that (X, Y) is a pair of spaces and Q is an R–module. Fix $k \in \mathbb{Z}$. Show that there is a natural isomorphism between the functors C and D where

$$(X, Y) \overset{C}{\mapsto} C^k(X, Y; Q) = \mathrm{Hom}_R(C_k(X, Y); Q)$$

and

$$(X, Y) \overset{D}{\mapsto} D^k(X, Y; Q) = \ker(C^k(X; Q) \to C^k(Y; Q))$$

Exercise 3.4. Suppose that X is a space and R is a commutative ring with unit. Let $\epsilon \in C^0(X; R)$ be the *augmentation homomorphism*: for all singular zero-simplices σ^0 we have $\epsilon(\sigma^0) = 1_R$. Now prove that the cup product at the level of cochains is:

- R–linear in both coordinates,
- associative, and
- has $\epsilon \in C^0(X; R)$ as its identity element.

Show, by means of an example, that the cup product at the level of cochains is not graded commutative.

Exercise 3.5.

- Prove that $H_2(S^1; \mathbb{Z}) \cong 0$.
- Let $\omega \in H^1(S^1; \mathbb{R})$ be the (class of the) winding cocycle. Prove that $\omega \cup \omega = 0$.
- [Challenge] Give *direct* proofs of the above: that is, from the definitions.
- [Challenge] More generally, give a direct proof that $H_k(S^1, \mathbb{Z})$ vanishes for $k > 1$.

Exercise 3.6. Suppose that $(A, C), (B, D) \subset (X, Y)$ are pairs of spaces. Suppose that X is contained in the union of the interiors of A and B; similarly suppose that Y is contained in the union of the interiors of C and D. We call $\mathcal{U} = \{(A, C), (B, D)\}$ an *excisive cover* of the pair (X, Y). We define $C_k^{\mathcal{U}}(\cdot)$ to be the R–module of (relative) singular chains in the given (pair of) space(s) *subordinate* to the cover $\mathcal{U}$. We define $C_k^{\mathcal{U}}(X, Y)$ to be the cokernel of the inclusion $C_k^{\mathcal{U}}(Y) \to C_k^{\mathcal{U}}(X)$.

Fix Q, an R–module. Define $C^k_{\mathcal{U}}(\cdot) = \mathrm{Hom}_R(C_k^{\mathcal{U}}(\cdot); Q)$.

- Show that $C^k_{\mathcal{U}}(X, Y; Q)$ is naturally isomorphic to $D^k_{\mathcal{U}}(X, Y; Q) = \ker(C^k_{\mathcal{U}}(X; Q) \to C^k_{\mathcal{U}}(Y; Q))$.
- Show that $H^*_{\mathcal{U}}(X, Y; Q) \cong H^*(X, Y; Q)$.
- Prove the relative version of Meyer-Vietoris; that is, the following sequence is exact:

$$\ldots H^k(X, Y; Q) \to H^k(A, C; Q) \oplus H^k(B, D; Q) \to H^k(A \cap B, C \cap D; Q) \to H^{k+1}(X, Y; Q) \ldots$$