11/1/23 GEOMETRIC TOPOLOGY

L2 Another reference: Geometric Knot Theory
See webpage for class details!

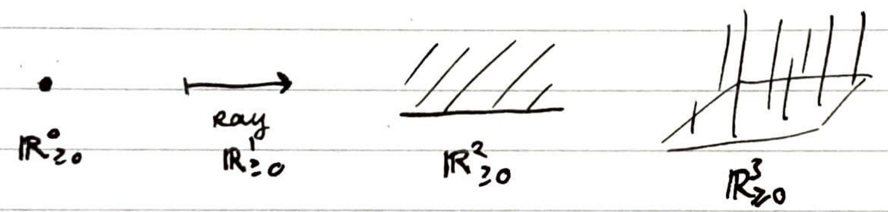
Challenge. Draw $\mathbb{RP}^2$ (in $\mathbb{R}^3$?).

MANIFOLDS WITH BOUNDARY

Defn. Suppose M is a topological space which is nonempty, and that
- M is Hausdorff
- every point $x \in M$ has a nbd. U so that $U \cong \mathbb{R}^n$ OR $U \cong \mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n \mid x_1 \geq 0\}$.

Then we call M an n-manifold with boundary.

Pictures



Q. $\mathbb{B}^n$ is an n-manifold with boundary, but not an n-manifold.

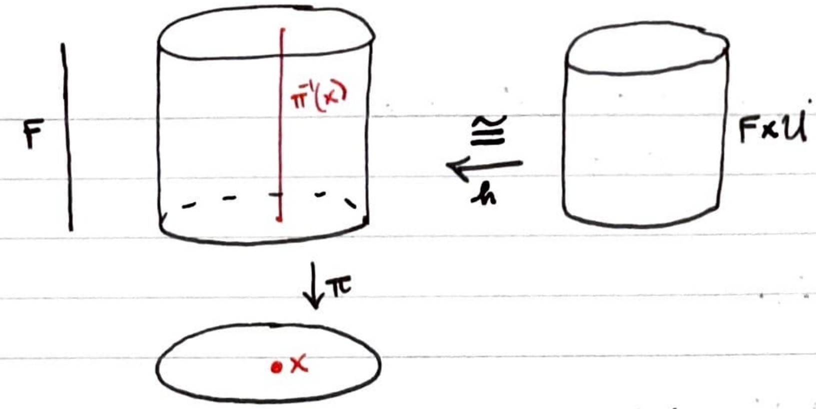
Defn. Let M be an n-manifold w/∂. Define
$$\partial M = \{x \in M \mid x \text{ has no } \mathbb{R}^n \text{ nbd.}\}$$
$$\mathring{M} = \operatorname{int}(M) = M \setminus \partial M$$

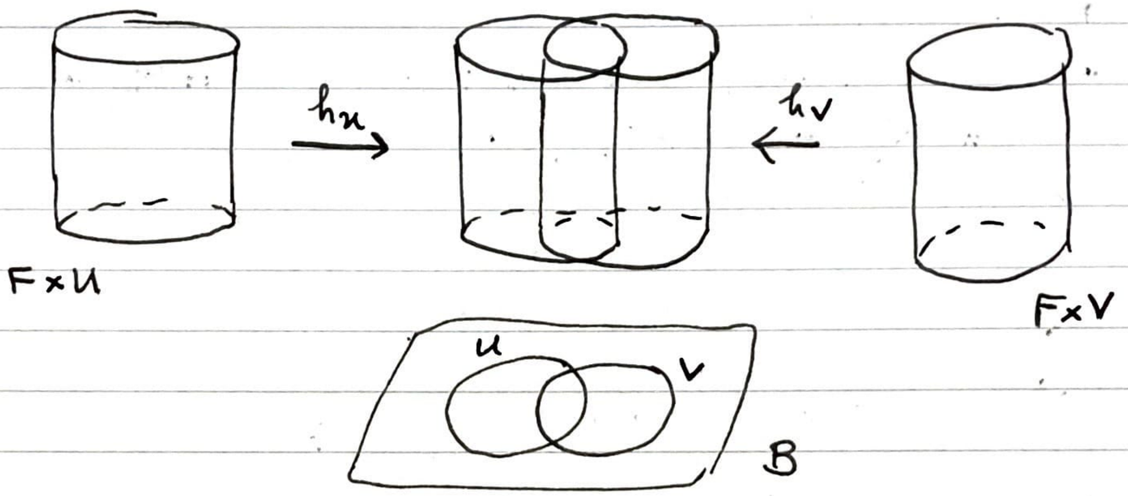
Q. $\partial \mathbb{B}^n = S^{n-1}$.

Q. If $\partial M \neq \emptyset$, then ∂M is an $(n-1)$-manifold, and $\partial\partial M = \emptyset$.

BUNDLES

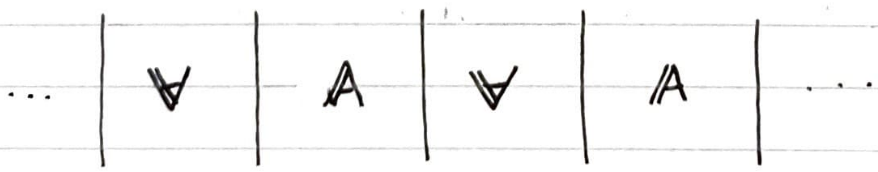
Defn. Suppose F, M, B are manifolds (possibly with ∂).
Suppose $\pi : M \to B$ is surjective.
We call $\pi : M \to B$ an F-bundle if
~~f~~ for all $x \in B$
- $\pi^{-1}(x) \cong F$
- there's a nbd. U of x in B so that
$$\pi^{-1}(U) \underset{h}{\cong} F \times U.$$
with $\pi \circ h = \pi_U$ (projection to the second factor).



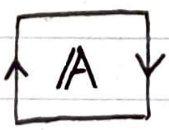


Note. $h_u^{-1}\circ h_v : F\times(U\cap V) \longrightarrow F\times(U\cap V)$ is an overlap map.
The homeomorphisms
$$h_u^{-1}\circ h_v\Big|_{h_v^{-1}(\pi^{-1}(x))}$$
as U, V, x vary form the structure group of the bundle.
Often, we restrict the structure group to consist of isometries.

Eg. $M^2 = \mathbb{R}\times I/(x,y)\sim(x+1,1-y)$.



Q. M^2 is a manifold with boundary.

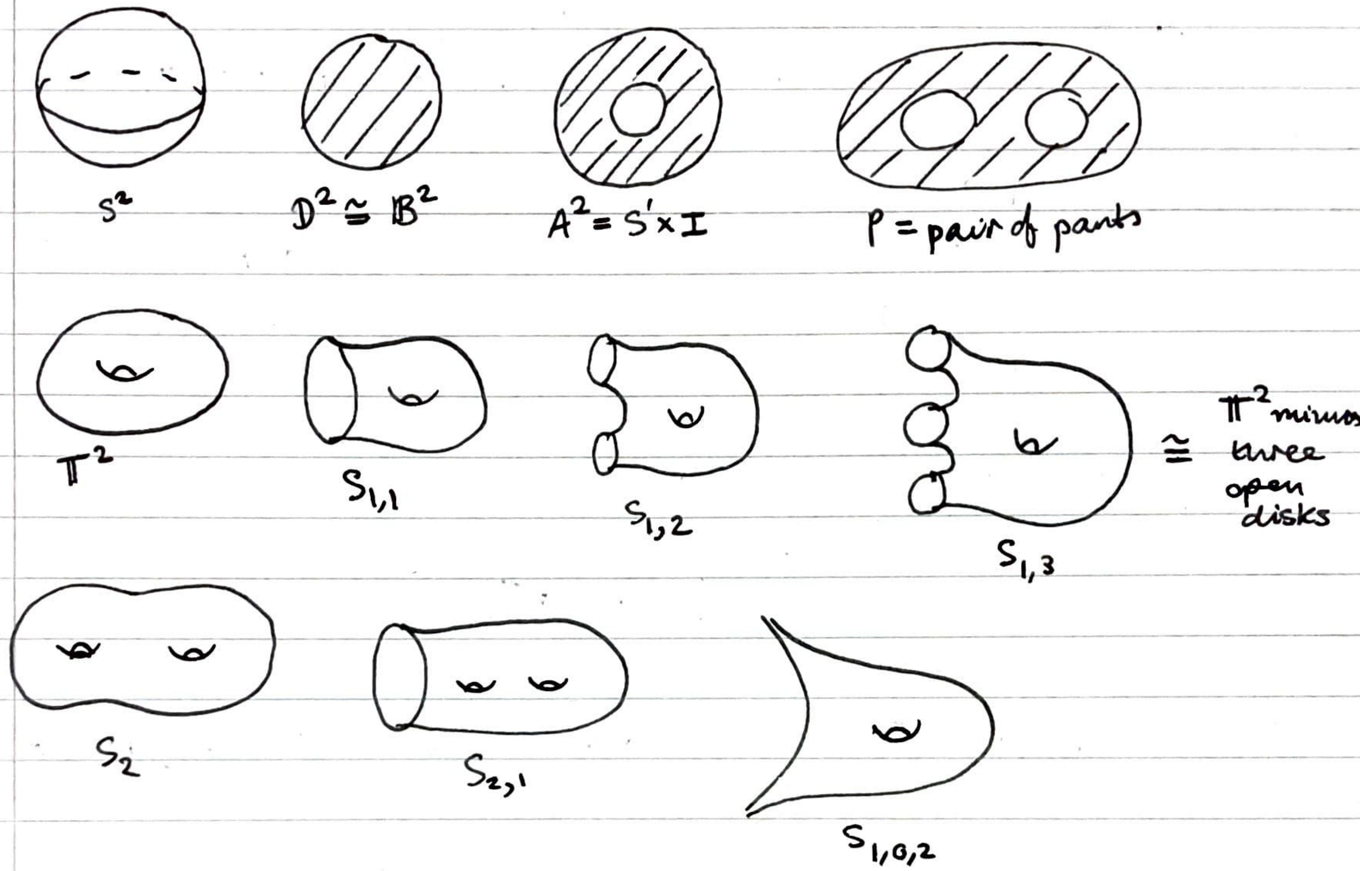
Q. $M^2 \cong I^2/(0,y)\sim(1,1-y)$ 

Q. M^2 is a non-trivial (non-product) I-bundle over S^1.
What's the structure group?

Q. $M^2 \cong$  , the ½-twisted band.

Q. $\mathbb{RP}^2 \cong M^2 \sqcup D^2 /$ glue boundaries by homeo.

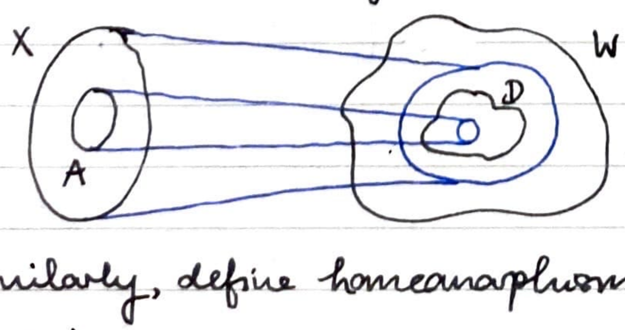
NAMES FOR SURFACES



LOCALLY FLAT AND TAME

Notation. If $B \subset A \subset X$ are spaces, write (X, A) for the pair and (X, A, B) for the triple of spaces.

Defn. $f : (X, A) \longrightarrow (W, D)$ a map of pairs if $f : X \to W$ is continuous and $f(A) \subset D$.



Similarly, define homeomorphisms, isotopies, homotopies of pairs.

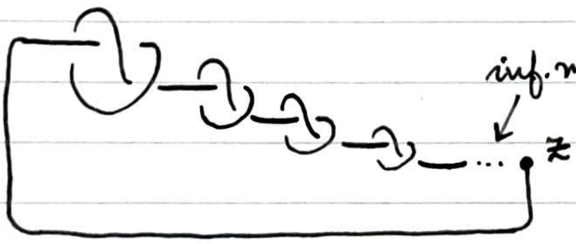
Suppose $N^n \xrightarrow{f} M^m$ is a map of manifolds which is injective [call this an embedding].

Defn. f is locally flat at $x \in f(N)$ if there is a nbd. U of x in M so that $(U, U \cap f(N), x) \cong (\mathbb{R}^m, \mathbb{R}^n \times \bar{0}, 0)$.

Eg. Trefoil

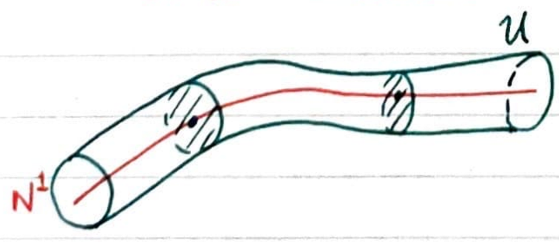
a locally flat map $f: S^1 \to \mathbb{R}^3$

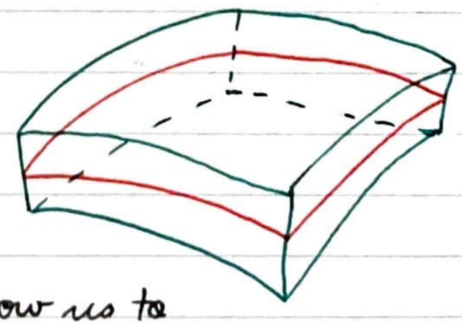
Challenge.



Prove that this map $S^1 \to \mathbb{R}^3$ is locally flat except at z.

Thm. Suppose M^m is an m-manifold ($m \leq 3$), and suppose $f: N^n \to M^m$ is locally flat.
Then f is tame, that is $f(N)$ has an open nbd. U which is a $\mathring{\mathbb{B}}^{m-n}$-bundle.





Point. Tubular neighbourhoods will allow us to perform various cut-and-paste operations on submanifolds.

(Hirsch, 1968) $\exists S^4 \hookrightarrow M^7$ which is locally flat but not tame.