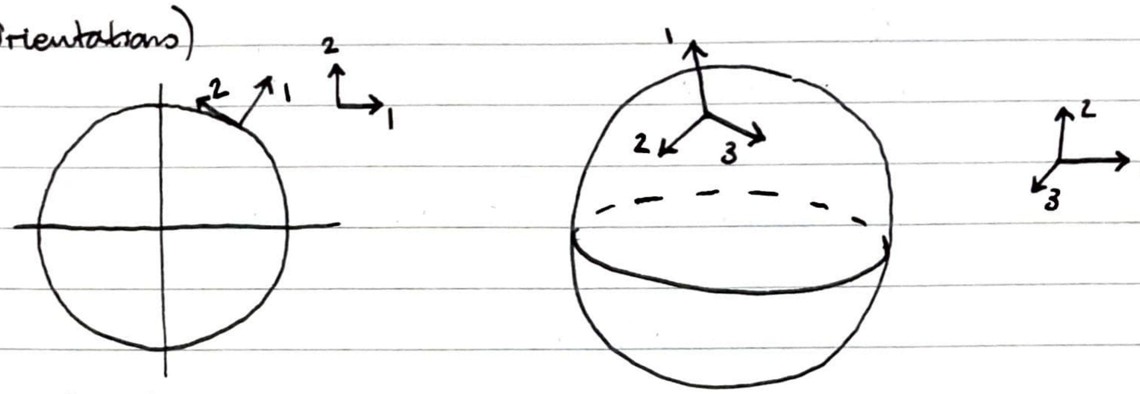
26/1/23 GEOMETRIC TOPOLOGY

CONNECT SUMS

Suppose M, N are n-manifolds (also oriented).
Suppose $D \subset M$ and $E \subset N$ are nicely embedded (closed) n-balls.
We equip ∂D and ∂E with their induced orientations.

Pictures. (Orientations)



$D^2 \subseteq \mathbb{R}^2$

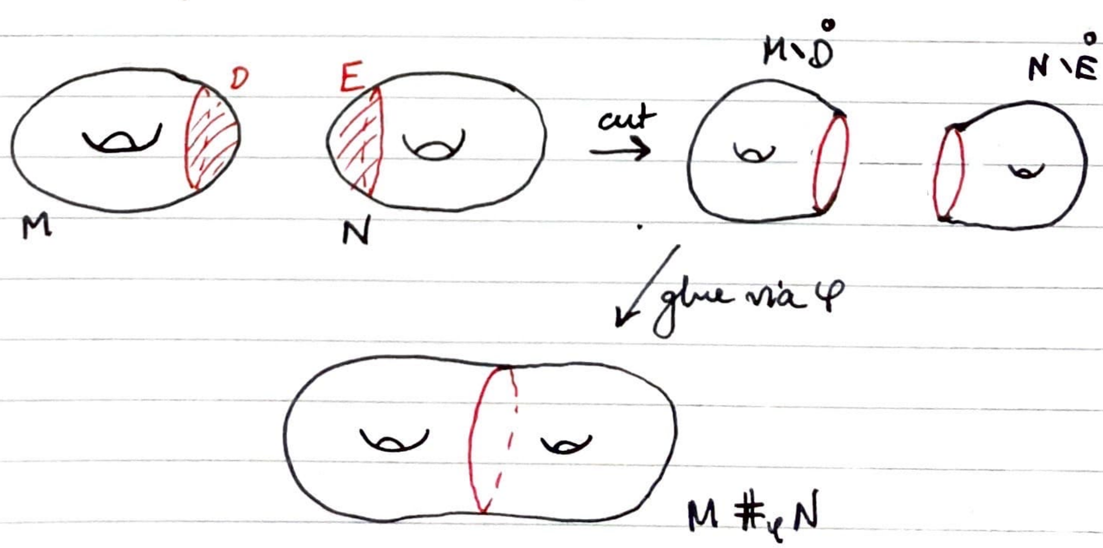
That is, move the given frame $(\sigma_i)_{i=1}^{n}$ so that σ_1 agrees with the outward normal to ∂D.
The vectors $(\sigma_i)_{i=2}^{n}$ gives a frame for the $(n-1)$-manifold ∂D, and so orients it.

Q. Define induced orientation of an $(n-1)$-mf. in an n-mf. in the topological category.

Pick an orientation-reversing homeomorphism (diffeomorphism) $\varphi : \partial D \to \partial E$.

Defn. The connect sum of M and N along φ is
$$M \#_{\varphi} N = (M \setminus \mathring{D}) \sqcup (N \setminus \mathring{E}) / \varphi$$

Pictures.



Q. $M \#_\varphi N$ inherits an orientation from M, N which agrees with induced orientation on $M \setminus \mathring{D}$ and $N \setminus \mathring{E}$.

Q. Prove $M^n \# S^n \cong M^n$, $n = 1, 2, 3$.

Q. Suppose M, N are compact, connected, oriented n-manifolds. Fix D, E, φ as above.
Compute $\pi_1(M \#_\varphi N)$ in terms of $\pi_1(M)$ and $\pi_1(N)$.

Q. Likewise, compute $H_k(M \#_\varphi N)$.

Q. Find all covers of $\mathbb{RP}^3 \# \mathbb{RP}^3$.

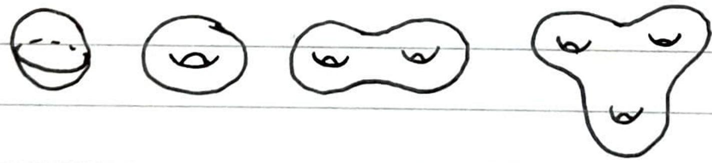
Defn. The *surface S_g of genus g* is defined by

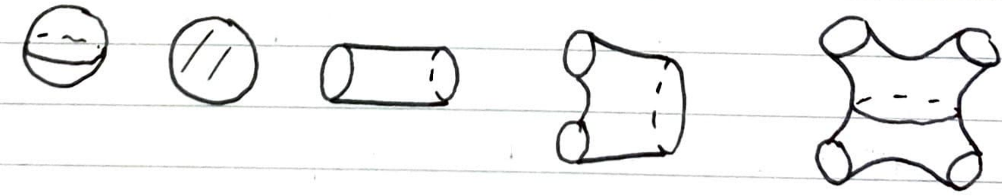
$$S_0 = S^2, \qquad S_{g+1} = S_g \# T^2.$$

Defn. The *planar surface* P_n is defined by

$$P_0 = S^2, \quad P_{n+1} = P_n \# D^2.$$

Pictures.





Goal of topological manifolds: classify them!

$\underline{n=0}$ point.

$\underline{n=1}$ circle

$\underline{n=2}$ S_g for $g \in \mathbb{N}$

$\underline{n=3}$ this module.

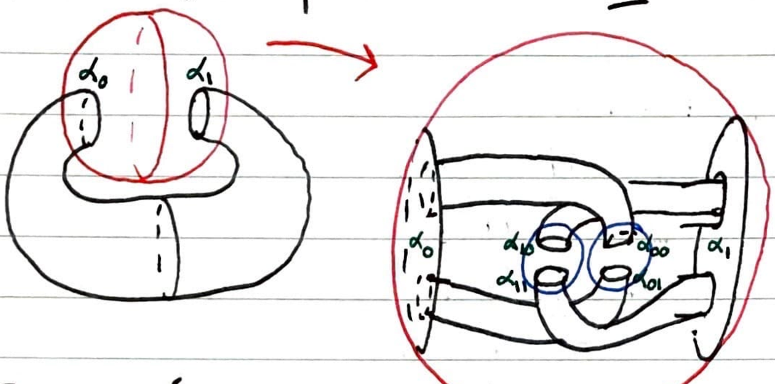
for connected, compact, oriented n-manifolds.

$n \geq 4$ Undecidable!

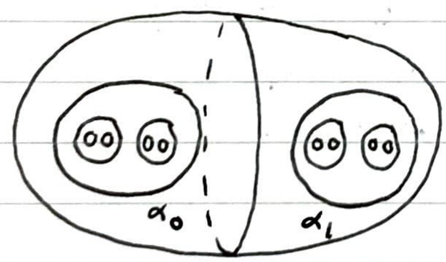
Moral. $\pi_1(M)$ "determines" the geometry, algebra, topology ... of low-dimensional manifolds, in dimensions ≤ 3.
This is very false in higher dimensions.

Q. Compute π_1 of S^4, $\mathbb{CP}^2$, $S^2 \times S^2$.
[These are not homeomorphic – homology; Poincaré duality.]

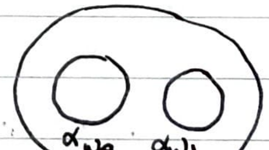
Picture. Alexander horned sphere – this is not "nicely" embedded.



The disks (with two smaller disks punched out) are labelled $\alpha_{i_0 i_1 \ldots i_n}$.



Inside α_W:



Q. The Alexander horned sphere is an embedding of S^2 into S^3 which is not locally flat along a Cantor set.

Indirect proof: let D, E be the two open components of $S^3 \setminus A$. Then D (inside) is homeo to $\mathbb{B}^3$ and $\pi_1(E)$ is infinitely generated.

Thm. (Alexander–Schoenflies) If $S^2 \hookrightarrow S^3$ is locally flat then there is a homeomorphism of S^3 making the 2-sphere round.

Thm. (Jordan–Schoenflies) For $S^1 \hookrightarrow S^3$.

Open. For diffeos. $S^3 \hookrightarrow S^4$. (The Smooth Schoenflies Conjecture in dim 4.)

Rk. The classification of surfaces relies on this theorem:

Thm. (Radu, 1910's?) Every surface admits a triangulation.

Now:

Thm. (Moise, 1952) Every 3-manifold admits a triangulation.

In higher dimensions: life is more complicated.

CONNECT SUMS OF PAIRS

Suppose (M, A) and (N, B) are manifold pairs, with $n = \dim(M) = \dim(N)$,

$b = \dim(A) = \dim(B)$.

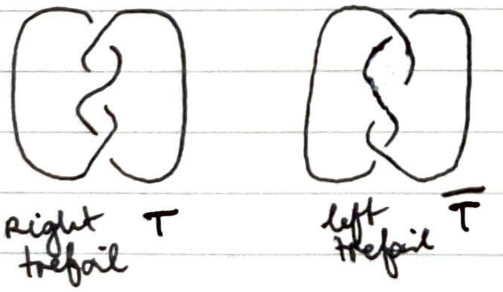
Pick nice $D \subset M$ and $E \subset N$ so that

$$(\mathring{D}, \mathring{D} \cap A) \cong (\mathbb{R}^n, \mathbb{R}^b)$$
$$(\mathring{E}, \mathring{E} \cap B) \cong (\mathbb{R}^n, \mathbb{R}^b)$$

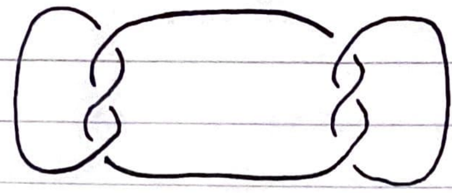
Choose $\varphi : (\partial D, \partial(D \cap A)) \longrightarrow (\partial E, \partial(E \cap B))$ an orientation-reversing homeomorphism, and glue:

$$(M, A) \#_\varphi (N, B) \cong (M \# N, A \# B).$$

Picture. $(n, b) = (3, 1)$.



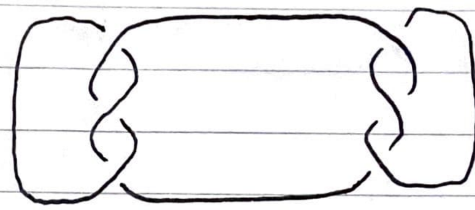
$(S^3, T) \# (S^3, T)$ gives



granny knot

Notation: $T \# T$.

$(S^3, T) \# (S^3, \overline{T})$ gives



reef (square) knot

Notation: $T \# \overline{T}$

Q. $\pi_1(X_{T\#T}) \cong \pi_1(X_{T\#\overline{T}})$,
i.e. the fundamental groups are isomorphic.

Q. $X_{T\#T}$ is not homeo. to $X_{T\#\overline{T}}$.