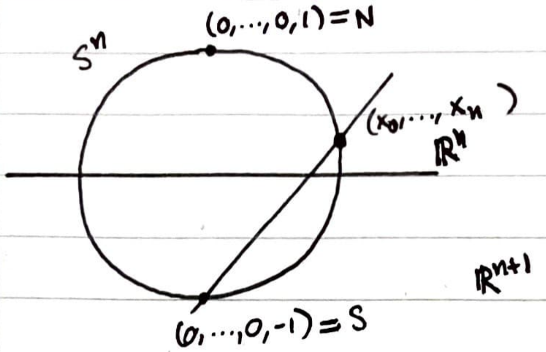
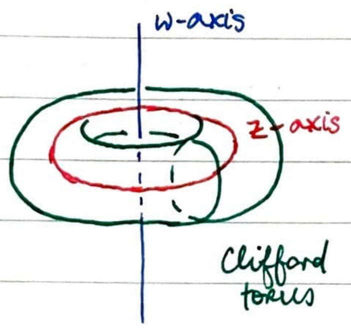
LECTURE 6

STEREOGRAPHIC PROJECTION



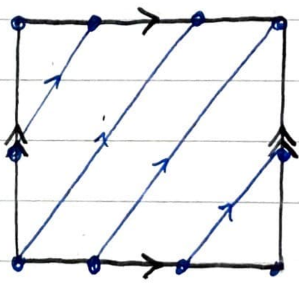
Suppose N, S are the "north" and "south" poles of S^n.
Define
$$st_S(x_0,\dots,x_n) = \frac{1}{x_n+1}(x_0,\dots,x_{n-1}).$$

Picture. For $n=3$.



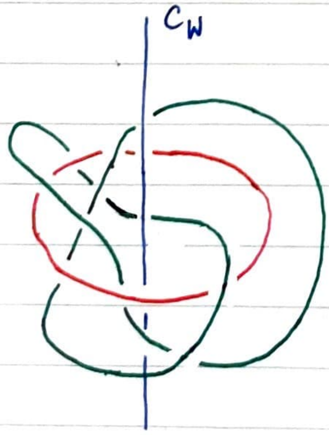
Set $C_W = \{(z,w) \in S^3 \mid z = 0\}$
$C_Z = \{(z,w) \in S^3 \mid w = 0\}$
$T = \{(z,w) \in S^3 \mid |z| = |w| = 1\}$

T "unwrapped"

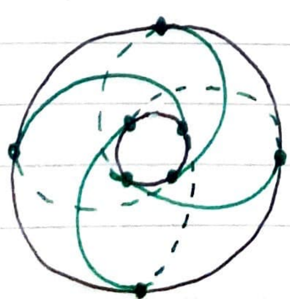


We draw the slope (p,q) in T by taking all the lines of slope p/q through the points k/p, $k = 0, 1, \dots, p-1$.

Picture.



Q. Draw a better picture of stereographic projection of $(3,2)$ torus knot into $\mathbb{R}^3$.



Bird's eye view of $(4,3)$.

Reference. Topologists Picture Book, by George Francis.

Q. The (p,q) and (q,p) torus knots (in S^3) are isotopic (possibly after reflection).

Q. Consider the action $S^1 \times S^3 \longrightarrow S^3$ defined via
$$e^{i\theta}\cdot(z,w) = (e^{qi\theta}z,\ e^{pi\theta}w).$$
This has, on T, orbits isotopic to the (p,q) torus knot.
Also C_z and C_w are "critical orbits" of the flow (that is, the action has kernel there).

Q. With this action, $S^3/S^1 = B_{p,q}$ is a two-sphere [and $c_w = [C_w]$, $c_z = [C_z]$ are cone points].

Q. The induced map
$$S^3 \setminus (C_w \cup C_z) \longrightarrow B_{p,q} \setminus \{c_w, c_z\}$$
is an S^1-bundle.

Rk. If $p,q = 1,1$ (or $\pm 1, \pm 1$), we get an (oriented, left/right) Hopf fibration of S^3.

the right-handed Hopf link.

Rk. Compute linking number of oriented two-component link L
$$\operatorname{lk}(L) = \frac{\#(\times) - \#(\times)}{2}.$$

Rk. (Hopf fibration stuff) There are two Lie groups acting nicely on S^3.
Real: since $S^3 \subseteq \mathbb{R}^4$ is a sphere, it is preserved by
$$SO(4) \cong \mathrm{Isom}^+(S^3).$$
Complex: since $S^3 \subseteq \mathbb{C}^2$, we "find" an action of $SU(2) \cong \widetilde{SO}(3) = \mathrm{Spin}(3)$ ($\widetilde{SO}(3)$ the universal (double) cover of $SO(3)$).

Morally. $SO(4) \cong \dfrac{\widetilde{SO}(3) \times \widetilde{SO}(3)}{\pm 1}$.

Quaternions. S^3 is a Lie group, the unit quaternions, so acts on itself on the left and right,

$$SO(4) \cong S^3 \times S^3 / \pm 1.$$

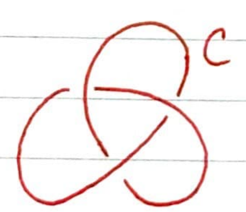
Q. $SO(3) \cong \mathbb{RP}^3$, so $\widetilde{SO}(3) \cong S^3$ (as Lie groups, not just as spaces).

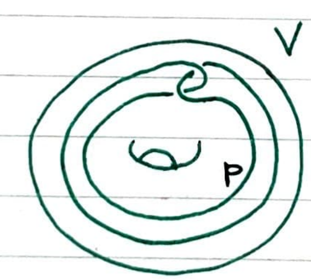
Q. The orbits of $S^1 \times S^3 \longrightarrow S^3$, $e^{i\theta}(z,w) = (e^{i\theta}z, e^{i\theta}w)$ are the cosets of a subgroup $S^1 < S^3$.

Ref. See Ch. 2 of Thurston's book.

SATELLITE KNOTS

Suppose $C \subset S^3$ is a knot, $V = S^1 \times D^2$ a solid torus, $P \subset V$ a knot.





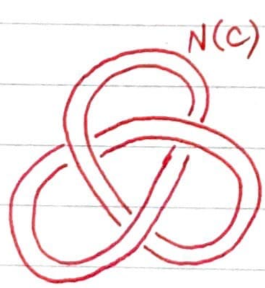
Fix some homeomorphism $\varphi : V \longrightarrow N(C)$ = a regular nbd. of C.

Call $\varphi(P) \subset S^3$ a s satellite knot with

(i) companion C

(ii) pattern (P, φ)

(iii) satellite torus $\partial N(C) = \varphi(\partial V)$.



— the twist comes from the choice of φ.

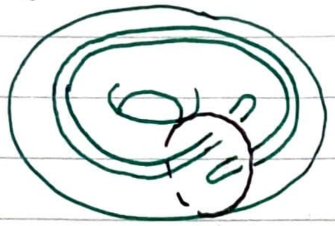
We ask for a few "non-triviality" hypotheses.

① C is not the unknot.

② P meets every meridian disk of V (including disks isotopic.

Q. Without both of these, every knot is a satellite knot.

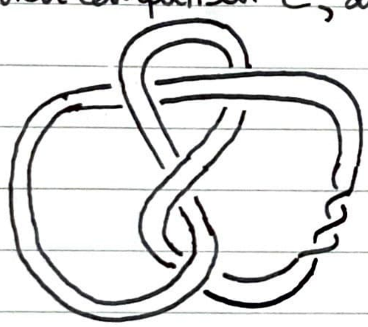
Rk. Every meridian disk is important:



this unknot meets "every" meridian disk, but the black disk can be isotopied to not meet the knot.

Q. A nontrivial connect sum $K \# K'$ is a satellite knot in two ways.

Defn. A knot $K \subset S^3$ is a cable along C if K is a satellite knot, with companion C, and $P \subset X$ is a torus knot.



the (2,3) cable of the figure 8 knot.

Conj. (Cabling Conjecture) Suppose $K \subset S^3$ is a knot. Suppose $r \subset \partial X_K$ is a slope. Suppose the Dehn filling $X_K(r)$ is reducible.
Then K is a cable and r is the annulus slope of the pattern knot P.