2/2/23 GEOMETRIC TOPOLOGY

LECTURE 7

UNIT TANGENT BUNDLES

Suppose M^n is a Riemannian manifold. The unit tangent bundle UTM of M is $UTM \subset TM$,

$$UTM = \{v \in TM \mid |v| = 1\}.$$

Q. (i) $UTS^1 \cong \mathbb{T}^2$
(ii) $UTS^2 \cong SO(3)$
(iii) $UT\mathbb{T}^2 \cong \mathbb{T}^3$
(iv) $UT\mathbb{E}^2 \cong S^1 \times \mathbb{R}^2$
(v) $UT\mathbb{H}^2 \cong S^1 \times \mathbb{R}^2$

Q. $SO(3) \not\cong S^1 \times S^2$

Rk. Suppose $X = \mathbb{H}^2/\Gamma$ is a compact, connected, oriented hyperbolic space, with $\Gamma \leqslant \mathrm{Isom}^+(\mathbb{H}^2) \cong PSL(2,\mathbb{R}) \cong UT(\mathbb{H}^2)$
Then $UTX \cong PSL(2,\mathbb{R})/\Gamma$,
and $\mathbb{H}^2 \cong {}_{S^1}\backslash PSL(2,\mathbb{R})$ and $X \cong {}_{S^1}\backslash PSL(2,\mathbb{R})/\Gamma$.

Here $S^1 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\} \cong SO(2)$.

Ref. See Geometries of Three-Manifolds, by Scott - BAMS.

Rk. If S is a geometric surface, then we have

$$\left.\begin{array}{c} S^1 \longrightarrow UTS \\ \downarrow \\ S \end{array}\right\} \text{is an } S^1\text{-bundle}$$

and the first example of Seifert fibred space.

Actions. Suppose G is a (Lie) group.
Suppose M is a (smooth) manifold.
Suppose $\rho: G \times M \to M$ is an action, i.e.

$$(g \cdot h) \cdot x = g \cdot (h \cdot x) \text{ for all } g, h \in G,\ x \in M.$$

Define. M/ρ (or M/G) is the quotient space.

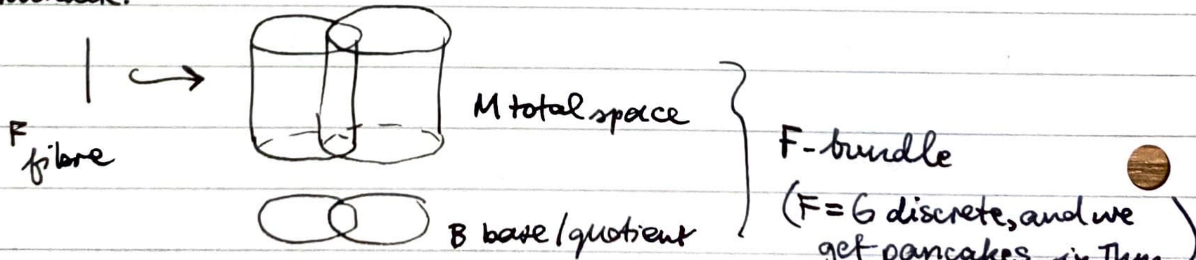
Defn. Say an action is continuous (or smooth) if $\rho_g \in \mathrm{Homeo}(M)$ (or $\mathrm{Diffeo}(M)$), for all $g \in G$.
Say an action is orientation-preserving if $\rho_g \in \mathrm{Homeo}^+(M)$, etc.

Defn. An action ρ is free if $g \cdot x = x$ iff $g = Id_G$.

Defn. An action ρ is properly discontinuous if for all compact $K \subset M$, we have $\{g \in G \mid (g \cdot K) \cap K \neq \emptyset\}$ is compact.
(If the topology on G is discrete then here compact iff finite.)

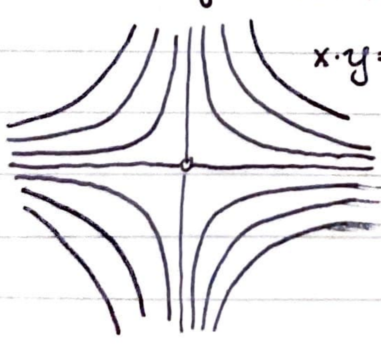
Thm. (Thurston, 3.5.7) Suppose G is discrete, acts smoothly, freely, properly discontinuously on M.
Then $M \to M/G$ is a covering of smooth manifolds.
(The fibre is isomorphic to G, $\left.\begin{array}{c} G \to M \\ \downarrow \\ M/G \end{array}\right\}$ is a G-bundle.)

Reminder.



Eg. $G \cong \langle g \rangle \cong \mathbb{Z}$ and $M = \mathbb{R}^2 \setminus \{(0,0)\}$, and $\rho: G \times M \to M$ defined by
$$g(x,y) = (2x, y/2) \quad \left(\text{the matrix } \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}\right).$$

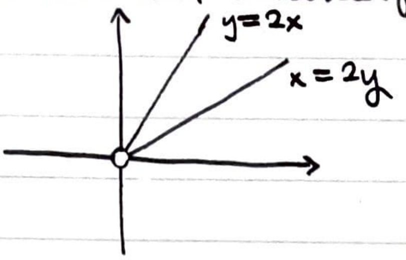
Q. The action of G on M is smooth and free, but not prop. discont.



Rk. The action is wandering: that is for all $x \in M$ there is an open $U \subset M$, $x \in U$, so that $\{g \in G \mid g \cdot U \cap U \neq \emptyset\}$ is finite.
[Check Thurston.]

Q. Describe M/G; NB. it is not Hausdorff! It's a 2-manifold, except that it's not Hausdorff.

Hint:



$y = 2x$ is taken to $x = 2y$.

ELLIPTIC MANIFOLDS

(Misha Kapovich article on proper discontinuity definition variants.)

Recall. $O(4) \cong \mathrm{Isom}(S^3)$, when S^3 is equipped with the round metric.

We equip $S^3 \subseteq \mathbb{R}^4$ with the round metric (from the metric on $\mathbb{R}^4$).

Q. (i) $\mathrm{Isom}(S^3) \cong O(4)$

(ii) $\mathrm{Isom}^+(S^3) \cong SO(4)$

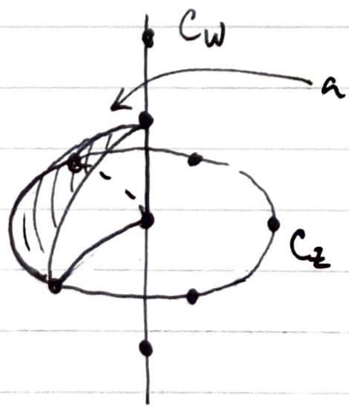
(iii) If $A \in O(4)$ and $\det(A) = -1$,
then A fixes a line in $\mathbb{R}^4$, and thus fixes a pair of pt.s of S^3.
(So orientation-reversing symmetries never act freely, and so can never be in the deck group of a space with universal cover S^3.)

Suppose $\Gamma \leq SO(4)$ acts freely on S^3, and that Γ is discrete.

Q./Defn. Then S^3/Γ is an elliptic manifold.

Eg. Fix $p \geq 1$ in $\mathbb{Z}$, let $\zeta_p = \zeta = \exp\left(\frac{2\pi i}{p}\right)$.
Define $\zeta . (z, w) = (\zeta . z, \zeta \cdot w)$.

So $L(p,1) = S^3/\langle \zeta_p \rangle$ is a manifold,
with $\pi_1(L(p,1)) \cong \mathbb{Z}/p\mathbb{Z} \cong \langle \zeta_p \rangle$.



a spherical tetrahedron with dihedral angles $(2\pi/5, 2\pi/4, 2\pi/4) \times 2$.

This is a Lens space.