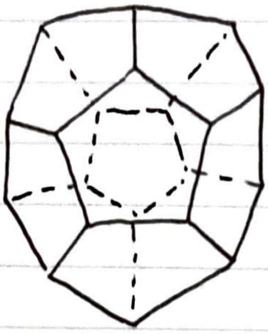
3/2/23 GEOMETRIC TOPOLOGY

LECTURE 8

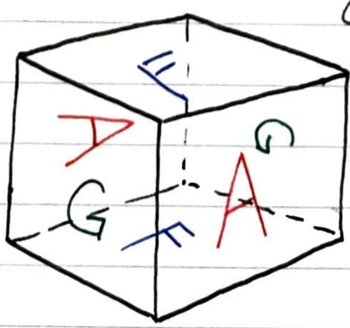
Eg.1



dodecahedron

glue opposite faces by a $\frac{1}{10}$ right-handed turn.
The quotient is PHS^3.

Eg.2



QT: the quarter-turn manifold; glue opposite sides with a $1/4$ right-handed turn.

References. A Topological Picture Book, by Francis.
The Shape of Space, by Weeks.

Plan ① Many examples ② Triangulations ③ Hyperbolic Dehn filling Thm.

Q. (i) PHS^3 and QT are three-manifolds.
(ii) Compute $\pi_1(QT)$ (and $\pi_1(PHS^3)$) $\rightsquigarrow$ compute H_1.

POINCARÉ HOMOLOGY SPHERE

Let $D \subset \mathbb{R}^3$ be the regular dodecahedron with vertices in S^2.
Let $\Gamma_D < SO(3)$ be the orientation preserving symmetries of D.
Recall that $S^3 \cong \widetilde{SO}(3)$.
That is, S^3 (a Lie group) is a double cover of $\mathbb{RP}^3 \cong SO(3)$.

Let $\Gamma_D^* < S^3 \cong SU(2)$ be the full preimage, also called the binary dodecahedral group.

Defn. $S^3/\Gamma_D^* \cong \mathbb{RP}^3/\Gamma_D$ is the Poincaré homology sphere.

Q. $PHS^3 \cong PHS^3$.

Moral. • If $\pi_1(X) = 1$ and G acts via isometries on X (freely, properly discont.) then X/G is a "view from above" of $X/G = M$. Useful information!

$$\begin{array}{c} X \\ \downarrow \\ X/G = M \end{array}$$

• On the other hand, suppose M is given to us as a triangulation, or a CW complex. This is a view from below. Basic information!

Point of geometrization: turn a view from below to the view from above [a v. flexible combinatorial description to the very rigid geometric description].

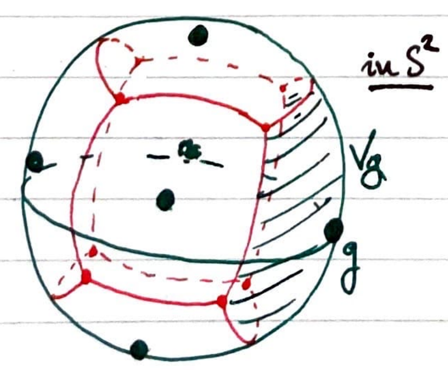
Q. Prove $H_k(PHS^3, \mathbb{Z}) \cong H_k(S^3, \mathbb{Z})$ for all k.
Poincaré conjectured that three manifolds are determined by their homology groups.
Show $\pi_1(PHS^3) \not\cong 1$, so Poincaré was wrong. But he is forgiven.

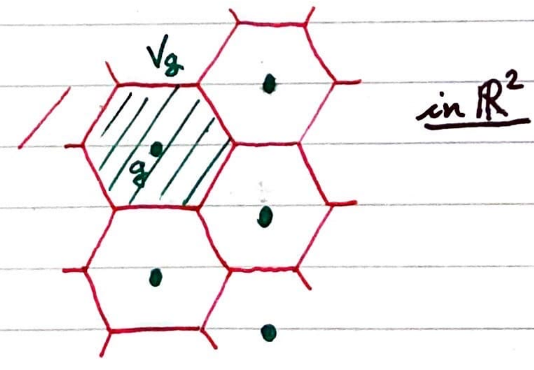
See. Gordon's article: Topology up to the 1960s.

Q. Take the Voronoi domains about the points of $\Gamma_D^* \subset S^3$.
This gives the cells of the boundary of the 120-cell, that is, 120 dodecahedra tiling S^3.
See sculpture.

Defn. Suppose X is a metric space. Suppose $G \subset X$ a discrete subset.
The set $V_g = \{x \in X \mid d(x,g) \leq d(x,h) \text{ for all } h \in G\}$
is called the Voronoi domain about g (relative to G).

Eg.

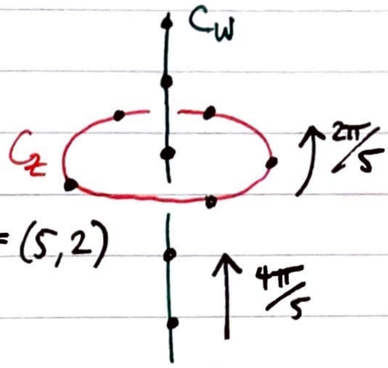




LENS SPACES

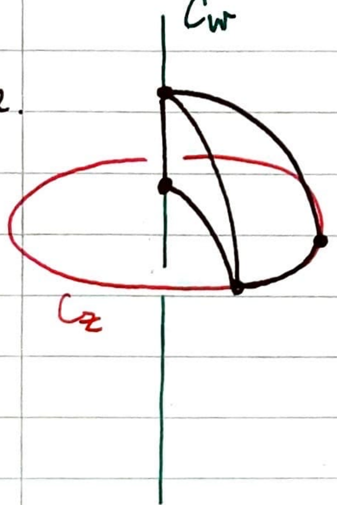
Fix $(p,q) = (1,1)$ or $p,q \in \mathbb{Z}$ so that $0 < q < p$ and $\gcd(p,q) = 1$.
Define $\zeta = \zeta_p = \exp\left(\frac{2\pi i}{p}\right)$, and define an action
$\rho^{p,q} = \rho : \langle \zeta_p \rangle \times S^3 \longrightarrow S^3$
by

$$\zeta \cdot (z,w) = (\zeta z, \zeta^q w).$$

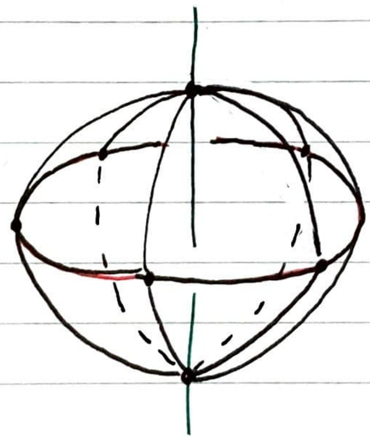


$(p,q) = (5,2)$

Picture.

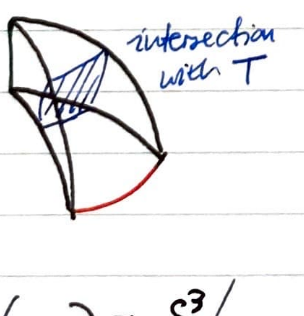


A lens is a collection of p such tetrahedra all sharing (say) an arc of C_w.



glue the bottom faces to the top by a $\frac{2\pi q}{p}$ rotation! (q clicks rotation)

Note. T = Clifford torus meets each tetrahedron in a "square".

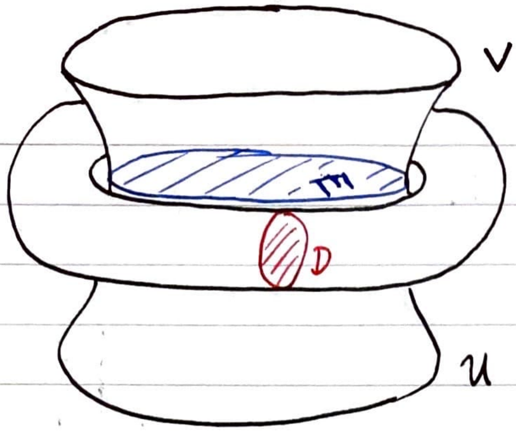


Let U, V be the closures of the complements of $S^3 \setminus T$.
These are solid tori.

Defn. $L(p,q) = S^3/\rho$ $[\rho = \rho^{p,q}]$ is the (p,q)-lens space.

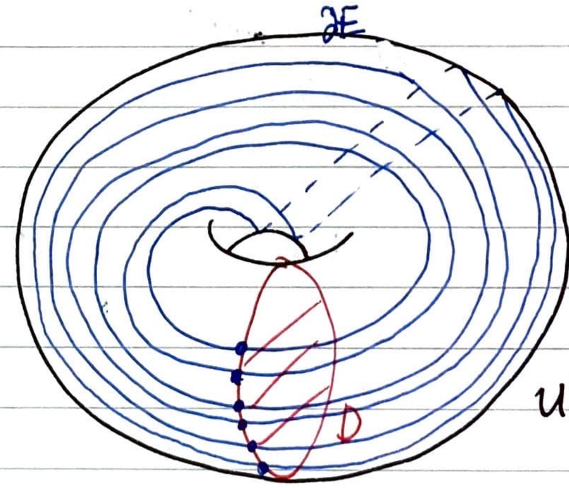
Q. Fix (p',q'). If $p' = p$ and $q' = \pm q^{\pm 1} \pmod p$ then $L(p',q') \cong L(p,q)$. [Hard: the converse also holds.]

Note. $T' = T/\rho$ is a two torus.
Also $U' = U/\rho$ and $V' = V/\rho$ are again solid tori.



D a merid. disk for U, E a merid disk for V.
Then D, E descend to U', V' to give D', E'.
Thus, $L(p,q) = U' \cup_{T'} V'$
is a decomposition of $L(p,q)$ into a pair of solid tori.

Q. Describe how $\partial D'$ and $\partial E'$ meet in T'.



Q. $L(1,1) \cong S^3 \cong SU(2) \cong \widetilde{SO}(3) \cong$ unit quaternions.
$L(2,1) \cong \mathbb{RP}^2 \cong SO(3) \cong UT(S^3) \cong PSU(2)$
$L(4,1) \cong UTP^2$

<u>Hard</u>. $L(7,1) \not\cong L(7,2)$

Q. $\pi_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$.

Reference. Spaces of Constant Curvature, by Wolf.