7/2/23 GEOMETRIC TOPOLOGY

LECTURE 9

## HANDLES

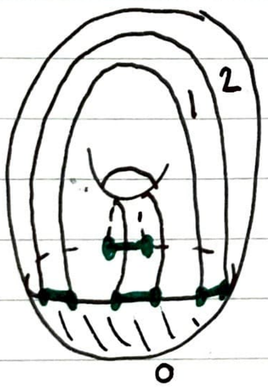
**Defn.** Fix $n \geq 0$. Fix $k$ with $0 \leq k \leq n$.
An *n-dimensional k-handle* is a copy of $\mathbb{B}^k \times \mathbb{B}^{n-k}$ $(\cong \mathbb{B}^n)$.
We call $\partial\mathbb{B}^k \times \mathbb{B}^{n-k}$ the *attaching region* for the k-handle.
We call $\mathbb{B}^k \times \partial\mathbb{B}^{n-k}$ the *attaching to region*.

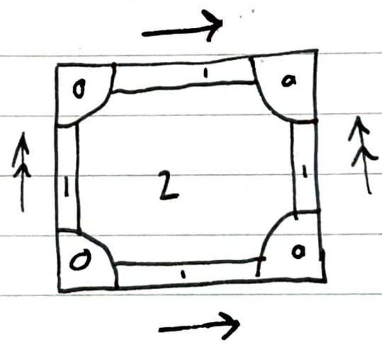
think of an n-diml. k-handle as a thickened k-cell! w/attaching map.

**Pictures.** $\underline{n=0}$  a point.

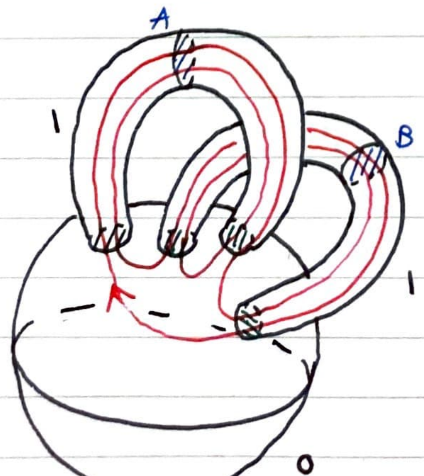
$\underline{n=1}$ $\underline{k=0}$  $\underline{k=1}$  attaching region

$\underline{n=2}$ $\underline{k=0}$  $\underline{k=1}$  $\underline{k=2}$ 

Eg. 

$\cong$ 


$\underline{n=3}$ $\underline{k=0}$  $\underline{k=1}$  $\underline{k=2}$  $\underline{k=3}$ 

Eg. 


The boundary of the zero-handle union the one-handles is a genus two surface with co-cores $A$ and $B$ for the one-handles.

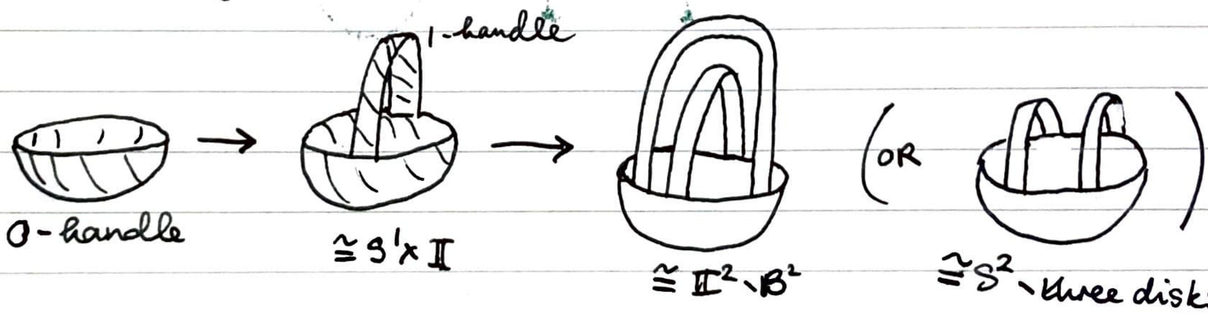
**Defn.** $B^k \times \{0\}$ is the **core** of $B^k \times B^{n-k}$.
$\{0\} \times B^{n-k}$ is the **co-core** of $B^k \times B^{n-k}$.

We may draw any curve $\alpha$ in the new "attach to" surface. Thicken this to get an annulus. We may attach a two-handle to this annulus.

**Q.** Suppose $M$ (a three-fold w/ boundary) has a handle structure with 1 zero-handle, 2 one-handles, 1 two-handle. List all possibilities for $\partial M$.

**Convention.** The attaching map should reverse the induced orientations.

Eg.

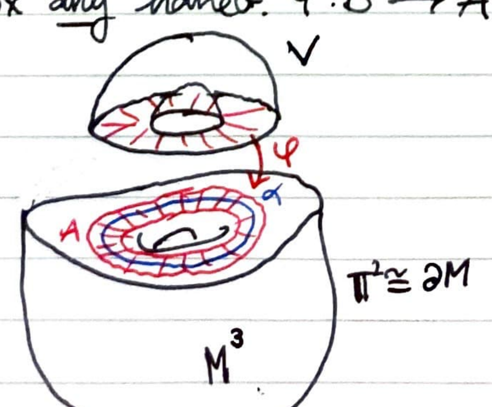



## DEHN FILLING

Suppose $M$ is a three manifold.
Suppose $\partial M = T^2$. Let $\alpha \subset T^2$ be a simple closed (essential) curve. Call $\alpha$ a **slope**.
Let $A = N(\alpha)$ be an annulus neighbourhood of $\alpha$ in $T^2$.

**Q.** Let $B^2 \times B^1 = V$ be a 2-handle with $B = \partial B^2 \times B^1$ its attaching region.
Fix any homeo. $\varphi : B \to A$.

**Q. [Crucial]**



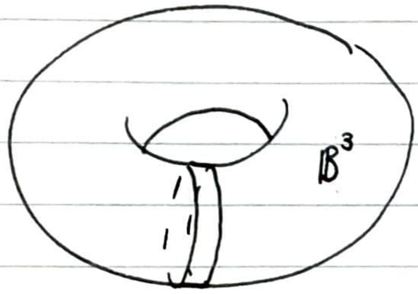

We form $M' = \dfrac{M \sqcup V}{\varphi}$.
$\partial M' \cong S^2$.

[Hint: classification of surfaces]

We thus immediately attach a 3-handle ($\mathbb{B}^3$) to obtain a manifold w/o boundary.

Q. If $M'$ is a three-manifold with $\partial M' \cong S^2$, then any pair of attaching maps of a 3-handle yield homeomorphic results.

Q. So we have $M'' = \left( \frac{M \cup V}{\varphi} \cup \mathbb{B}^3 \right)$ without boundary.
$V \cup \mathbb{B}^3$ in $M''$ is a solid torus.



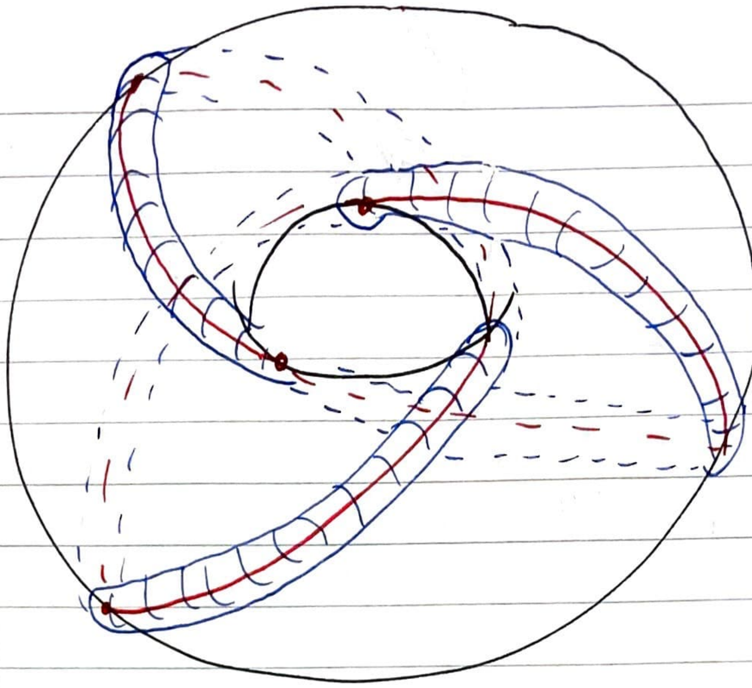

so two disks of $\partial \mathbb{B}^3$ are attached to $V$ and the remaining annulus of $\partial \mathbb{B}^3$ is attached to $T \setminus A \subset \partial M$.

Defn. $M''$ is the Dehn filling of $M$ along the slope $\alpha$.
We denote this by $M(\alpha)$.

Q. The result is independent of the choices made (other than $M$ and the isotopy class of $\alpha$ (which must be essential)).

Equivalently, we could start with $M$ and $\alpha$.
Define $U = \mathbb{D}^2 \times S^1$.
Pick any homeo. $\varphi : \partial U \to \partial M = T$ so that $\varphi(\partial \mathbb{D}^2 \times \{pt\}) = \alpha$. Then $M(\alpha) \cong M \sqcup U / \varphi$.
(The independence of choices is less clear.)

Q. Suppose $M^3 = \mathbb{D}^2 \times S^1$. List all (up to homeomorphism) three-manifolds obtained by Dehn filling $\partial M$.



Q: What manifolds do we get by drilling and filling torus knots, aka Dehn filling torus knot complements.