17/2/23 GEOMETRIC TOPOLOGY

LECTURE 10

ONE HALF LIVES, ONE HALF DIES

Suppose M is a connected compact oriented 3-manifold.
So ∂M is a disjoint union of oriented compact surfaces without boundary ($\partial\partial M = \phi$). So

$$\mathrm{rk}(H_1(\partial M; \mathbb{Z})) \text{ is } \underline{\text{even}}.$$

Define $\iota : \partial M \longrightarrow M$ to be the inclusion.
Define $\iota_* : H_1(\partial M; \mathbb{Z}) \longrightarrow H_1(M; \mathbb{Z})$
to be the induced homomorphism.

L. $\mathrm{rk}(\ker(\iota_*)) = \frac{1}{2}\mathrm{rk}(H_i(\partial M; \mathbb{Z}))$.

Rk. If A is a $\mathbb{Z}$-module, then $\mathrm{rk}(A)$ is the min. number of generators required to generate the torsion free part of A.

Rk. For us, both $H_1(\partial M)$ and $\ker(\iota_*)$ are torsion free.

Q. Pf. [Exercise [Hint: Poincaré duality for manifolds w/ boundary].

HOMOLOGICAL LONGITUDE

Suppose, additionally on M, that $\partial M \cong \mathbb{T}^2 = S^1 \times S^1$.
So $H_1(\partial M) \cong \mathbb{Z}^2$.

Eg. $M = X_K$ is a knot exterior in S^3.

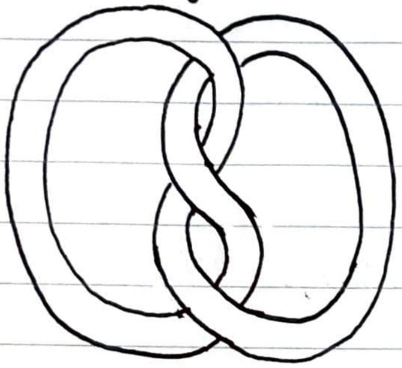
Then $\ker(\iota_*) \cong \mathbb{Z}$ is generated by "the" homological longitude. [There are two, but they are negs of each other.]
We denote this by $\lambda \subset \partial M$ a simple closed essential curve.

Note. Since $\lambda \in \ker(\iota_*)$, λ bounds a relative 2-cycle in M.

Thm. λ bounds a proper embedded oriented surface $(F, \partial F) \subset (M, \partial M)$ with $\partial F = \lambda$.
[with orientations.]

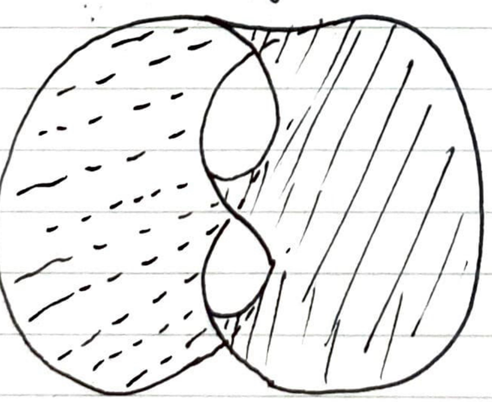
Q. Pf. [Exercize.

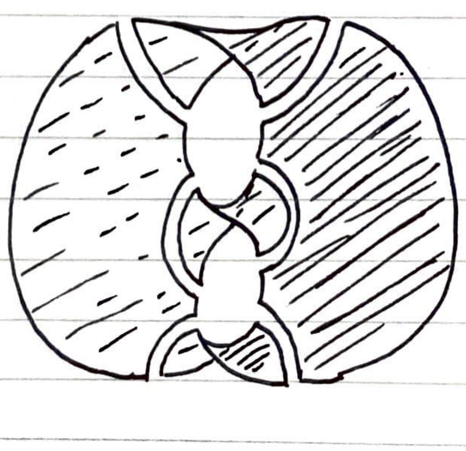
Eg. K the trefoil in S^3. Let $\mathcal{U} = N(K)$ be a reg. nbd.,
$X_K = S^3 \setminus \mathrm{int}(\mathcal{U})$,
$\partial X_K = \partial \mathcal{U} = T \cong \mathbb{T}^2$.



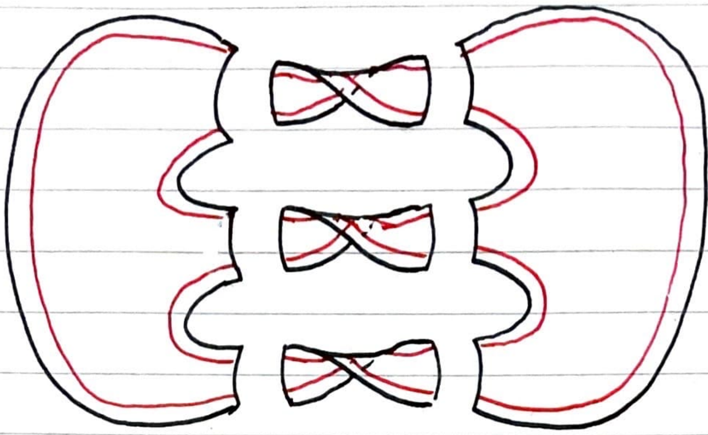
Q. Every knot $K \subset S^3$ bounds an embedded, oriented, connected surface F_K. This surface is called a Seifert surface for K. [Sometimes require F_K to be incompressible.] Say F_K spans K.

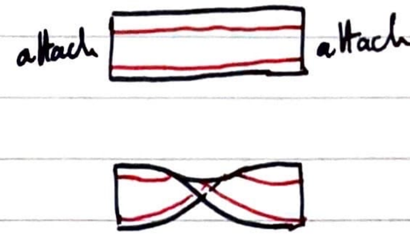
Defn. $g(K)$ = knot genus (= three-genus)
$= \min\{\text{genus}(F) \mid F \text{ spans } K\}$.





Half-twisted band

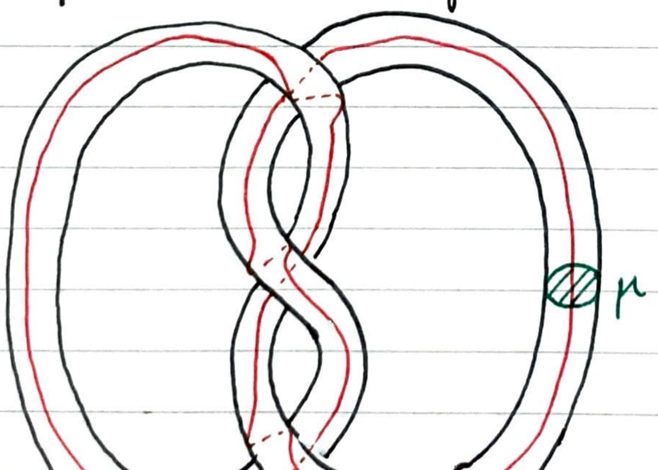


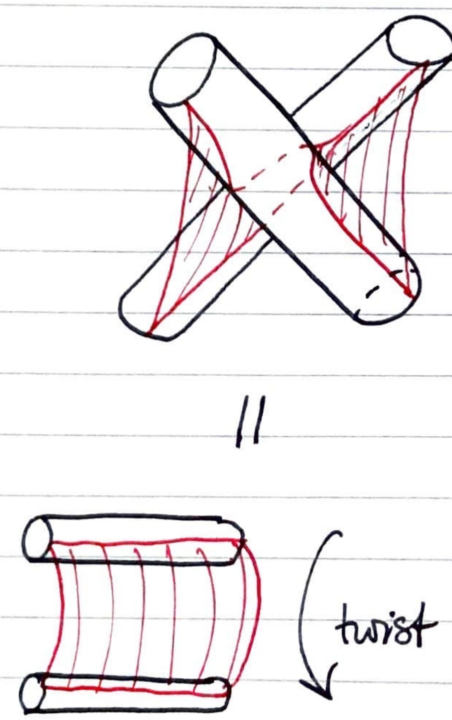


Let λ' be the pushoff of K in the direction given by the Seifert surface.
We arrange matters so that λ' lies in ∂U.

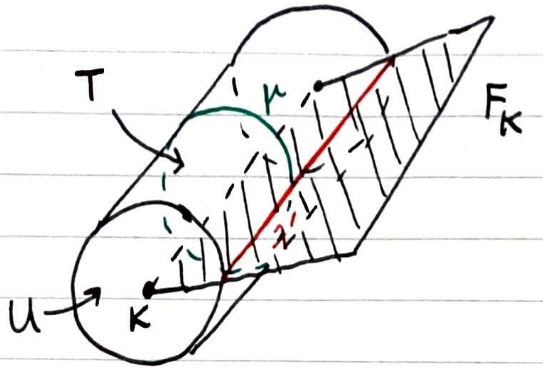
L. λ is isotopic to λ'.

Define μ to be $\partial \mathbb{D}^2 \times \{pt\}$ where $U \cong \mathbb{D}^2 \times S^1$.
Call μ the meridian of K.





Q. Above we pushed the "normal" to K determined by F (spanning surface). This gives λ'.
Show that pushing in the negative of the normal direction gives λ'' in ∂U which is isotopic to λ'.



cross-section.

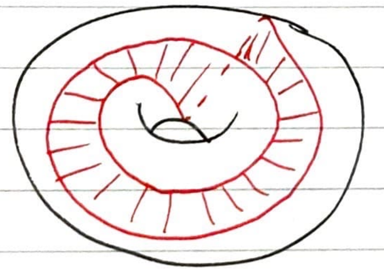
connected, $\boxed{\text{two-sided}}$

Aside. Suppose $(F, \partial F) \subset (M, \partial M)$ is properly embedded (∂'s could be $\emptyset$).

Defn. We say F is algebraically incompressible if the induced map on π_1 is injective.

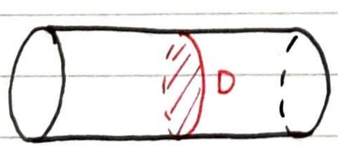
Defn. $F^2 \subset M^3$ is two-sided if $N(F) \cong F \times I$.

Eg.



The $(1,2)$ curve bounds a Möbius band.
This is one-sided.

Defn. Suppose $(F, \partial F) \subset (M, \partial M)$ as above. Suppose $(D, \partial D) \subset (M, F)$ has $D \cap F = \partial D$.



Call D a surgery disk.
Call D a compressing disk if ∂D is essential in F.

Defn. $(F, \partial F)$ in $(M, \partial M)$ is geometrically incompressible if F admits no compressing disks.

Thm. (Disk Theorem) Suppose F is two-sided.
Then F is algebraically incompressible iff F is geometrically incompressible.

Q. One direction is easy. Find and prove it.

Q. Show that the two-sided hypothesis is necessary.

Q. Suppose F^2 (closed, connected) lies in S^3.
<u>then</u> (i) F is two-sided
(ii) F is orientable
(iii) $F \cong S^2$ or F is compressible.