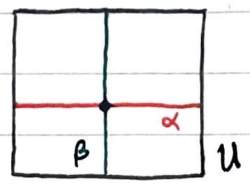
LECTURE 12

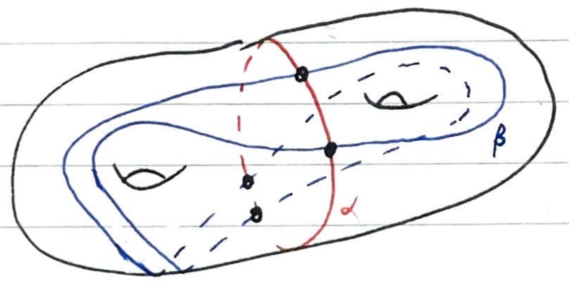
ALGEBRAIC INTERSECTION NUMBER

Suppose S is a connected, compact, oriented surface.
Suppose α, β are oriented simple closed curves.
Suppose α and β are transverse, i.e. for every point $x \in \alpha \cap \beta$, we have a nbd. $\mathcal{U}$ of x in S where we see:



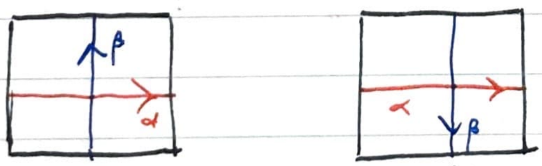
That is, x is sent to $(0,0)$,
$\alpha \cap \mathcal{U}$ is sent to $\mathbb{R} \times \{0\}$,
$\beta \cap \mathcal{U}$ is sent to $\{0\} \times \mathbb{R}$.

Picture.



$|\alpha \cap \beta| = 4$

We have two possibilities up to orientation-preserving homeo.

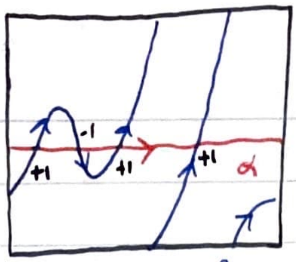


Defn. $\alpha \cdot \beta = \#\left(\begin{smallmatrix}\text{positive}\\ \text{intersections}\end{smallmatrix}\right) - \#\left(\begin{smallmatrix}\text{neg.}\\ \text{int.}\end{smallmatrix}\right)$, the algebraic intersection number.

Q. If α, β are transverse, then $|\alpha \cap \beta| < \infty$.
[Hint: α is compact.]

Q. $\alpha \cdot \beta = \alpha^*(\beta)$, where $\alpha \in H^1(S, \mathbb{Z})$ is the Poincaré dual to $[\alpha]$.

Q. $\alpha \cdot \beta$ is invariant of isotopy.

Q. 

$\alpha \cdot \beta = 2$

Thm. Suppose $S = \mathbb{T}^2$, μ, λ is a framing of $\mathbb{T}^2$.
Suppose α, β are slopes with
$$\alpha = p\mu + q\lambda, \quad \beta = r\mu + s\lambda \quad [p, q, r, s \in \mathbb{Z}].$$
Then
$$\alpha \cdot \beta = \det\begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Thm. (Moser, 1971) Suppose $K = K_{p,q}$ is a torus knot.
Suppose $r, s \in \mathbb{Z}$ with $\gcd(r, s) = 1$.
Define $\sigma = \det\begin{pmatrix} pq & r \\ 1 & s \end{pmatrix}$ [that is, the algebraic intersection ~~[illegible]~~ number between the annulus slope $\alpha = p \cdot q \cdot \mu + \lambda$ and the filling slope $\beta = r\mu + s\lambda$].
Then
(1) If $|\sigma| \geq 2$ then $X_K(r/s)$ is Seifert feibered over $S(a, b, c)$
(2) If $|\sigma| = 1$ then $X_K(r/s) \cong L(|r|, sq^2)$
(3) If $|\sigma| = 0$ then $X_K(r/s) \cong L(q, p) \# L(p, q)$.

Thm. (Dehn, 1910) Deals (sort of) with Dehn fillings of $K_{3,2}$ trefoil.
[Actually focuses on $X_K(1/n)$.]

Q. Using Moser or otherwise, show $\pi_1(X_K(1/n))$ is infinite for $n \neq 0, 1$.

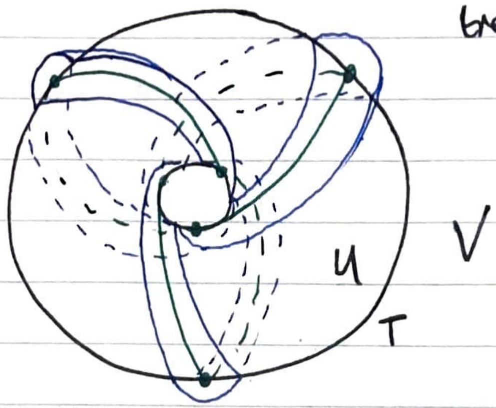
Q. $X_{K_{3,3}}(1/1) \cong PHS^3$ (Poincare homology S^3).

Long story about Poincaré, Heegaard, Dehn and etc!

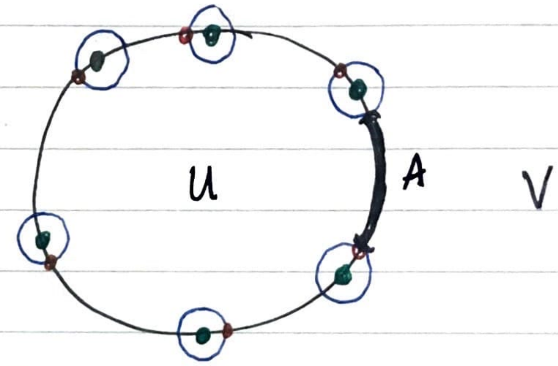
Pf. When $\sigma = 0$.
That is, $\det\begin{pmatrix} pq & r \\ 1 & s \end{pmatrix} = 0 = pqs - r$.
$\Rightarrow pqs = r$.

But $\gcd(r,s) = 1 \Rightarrow \quad s = \pm 1, \; r = \pm pq$.
Since β is a slope, may assume $s = 1$, $r = pq$.
[That is, $\beta = \alpha$ is the annulus slope]
[This is <u>also</u> called the Seifert fibre slope.]

trefoil $K_{3,2}$



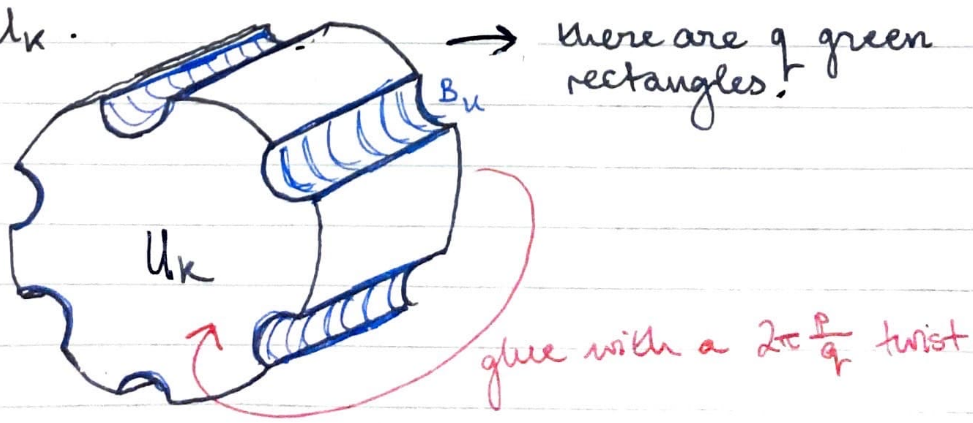
<u>Cross section</u>



<u>Notation.</u> T = Clifford torus —
U, V are closures of compt.s of $S^3 \setminus T$.
— = $N(K)$ = neighbhd. of knot K = —
— $\alpha = \beta$ is <u>the filling slope</u>.
Define $A = \overline{\partial T - N(K)}$
$U_K = \overline{U - N(K)}$
$V_K = \overline{V - N(K)}$

<u>Note.</u> $U_K \cup_A V_K = X_K$

<u>Picture</u>. of U_K.



$\rightarrow$ there are q green rectangles!

Let $W = D^2 \times S^1$ be a solid torus.

Define $W_\mathcal{U} = D^2 \times \{e^{i\theta} \mid 0 \leq \theta \leq \pi\}$.

$W_V = D^2 \times \{e^{i\theta} \mid \pi \leq \theta \leq 2\pi\}$.

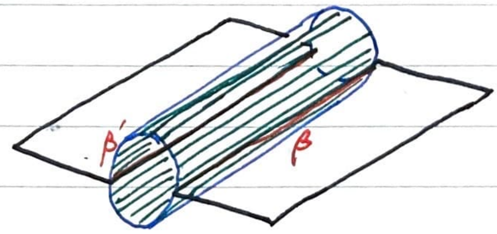
These are two-handles.

<u>Define</u> $B_\mathcal{U} = \partial N(K) \cap \mathcal{U}$

$B_V = \partial N(K) \cap V$

<u>Now</u>: $X_K(\beta) = X_K \cup_\beta W$

Picture of $N(K)$:



Note $A \cap \partial N(K)$

$= \beta \cup \beta'$,

parallel slopes.

So we foliate $B_\mathcal{U}, B_V$ by curves parallel to β, β'.

Now glue $\left\{\begin{array}{l} W_u \text{ to } \mathcal{U}_K \\ W_V \text{ to } V_K \end{array}\right\}$ along these copies of β.

<u>So</u>. $\mathcal{U}_K \cup B_u \cong L(q,p) - \mathrm{int}(B^3)$

$V_K \cup B_V \cong L(p,q) - \mathrm{int}(B^3)$

<u>So</u>. $X_K(\beta) = (\mathcal{U}_K \cup B_u) \cup_{S^2} (V_K \cup B_V)$

$\cong L(q,p) \# L(p,q)$.

$\llcorner$

Defn. On $S = S_1 = \mathbb{T}^2$, if $\alpha \cdot \beta = \pm 1$, we say the slopes α, β are Farey neighbours.

Q. Draw the Farey graph, where vertices are points of $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\frac{1}{0}\}$ and edges connect Farey neighbours.

Q. Do the other cases of Moser's Thm.
[Hint: if $\sigma = \pm 1$, then we are performing a Dehn twist between U and V (c.f. Lickorish).]

PROPERTY P

Defn. Suppose $K \subset S^3$ is a knot. Say K has property P [Bing] if no Dehn filling of K is a counterexample to the Poincaré conjecture [that is, if M closed, connected and $\pi_1(M) \cong 1$, then $M \cong S^3$].

Perelman proves Geometrisation and thus the Poincaré Conjecture.

But, anyway, Moser proves Property P for torus knots.
Someone proves it for satellite knots, for alternating knots, etc.

Suppose $K \neq U$ (the unknot).

Thm. (Gordon - Luecke) If $X_K(r/s) \cong S^3$ then $r/s = 1/0$. [Combinatorial topology]

Q. $H_1(X_K(r/s)) \cong \mathbb{Z}/r\mathbb{Z}$ (so the only case of interest is $r = 1$).

Thm. (Cyclic Surgery Theorem, Culler - Gordon - Luecke - Shalen)
If $r/s \neq \pm 1/1$, then $\pi_1(X_K(r/s)) \neq 1$. [Combinatorial topology + representation theory]

Thm. (Kronheimer - Mrowka) $\pi_1(X_K(1/1)) \neq 1$. [Gauge theory]

Q. $X_K(-1/1) \cong X_{\overline{K}}(1/1)$.

These theorems also prove Property P and each other in various combinations.

Q. $X_u(0/1) \cong S^2 \times S^1$.

PROPERTY R (Gabai, Gordon-Luecke, others)

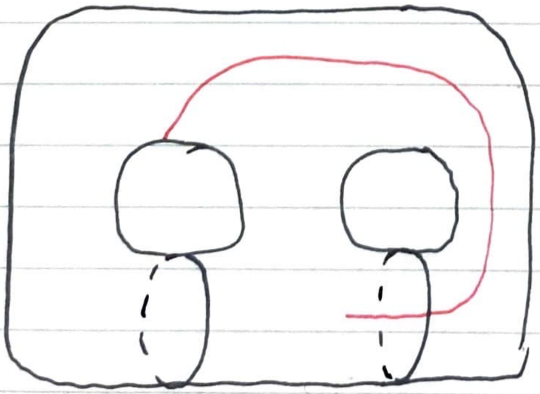
Suppose $K \subset S^3$ is a knot, $K \ncong U$.
Then $X_K(0/1) \ncong S^2 \times S^1$. [Uses theory of taut foliations.]

Q. $X_K(0/1)$ is a homology $S^2 \times S^1$!

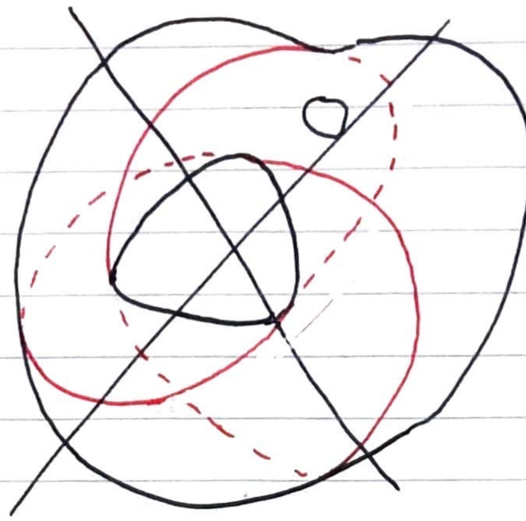
Conj (Cabling Conjecture) Suppose $K \subset S^3$ is a knot.
Suppose $X_K(r/s)$ is reducible
Then K is a cable knot [satellite with torus knot pattern], and r/s is the annulus slope.

Conj (Berge Conjecture) Suppose $X_K(r/s)$ is a lens space.
Then K is a Berge knot, that is K lies on the genus two Heegaard splitting of S^3 and is doubly primitive there.

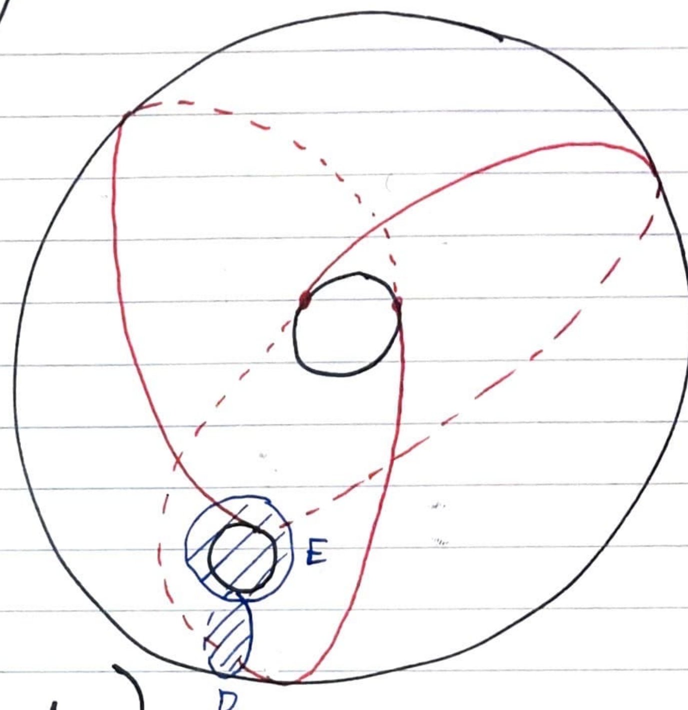
Picture.



Wrong:

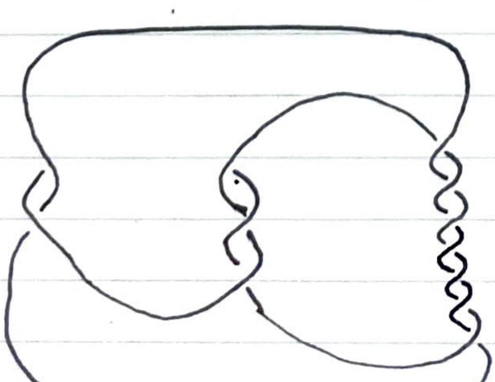


Trefoil knot in U (handlebody) with disks D, E showing it is doubly primitive ($|D \cap K| = |E \cap K| = 1$, $D \subset U$, $E \subset V$, $K \subset U \cap V$ the Heegaard surface).

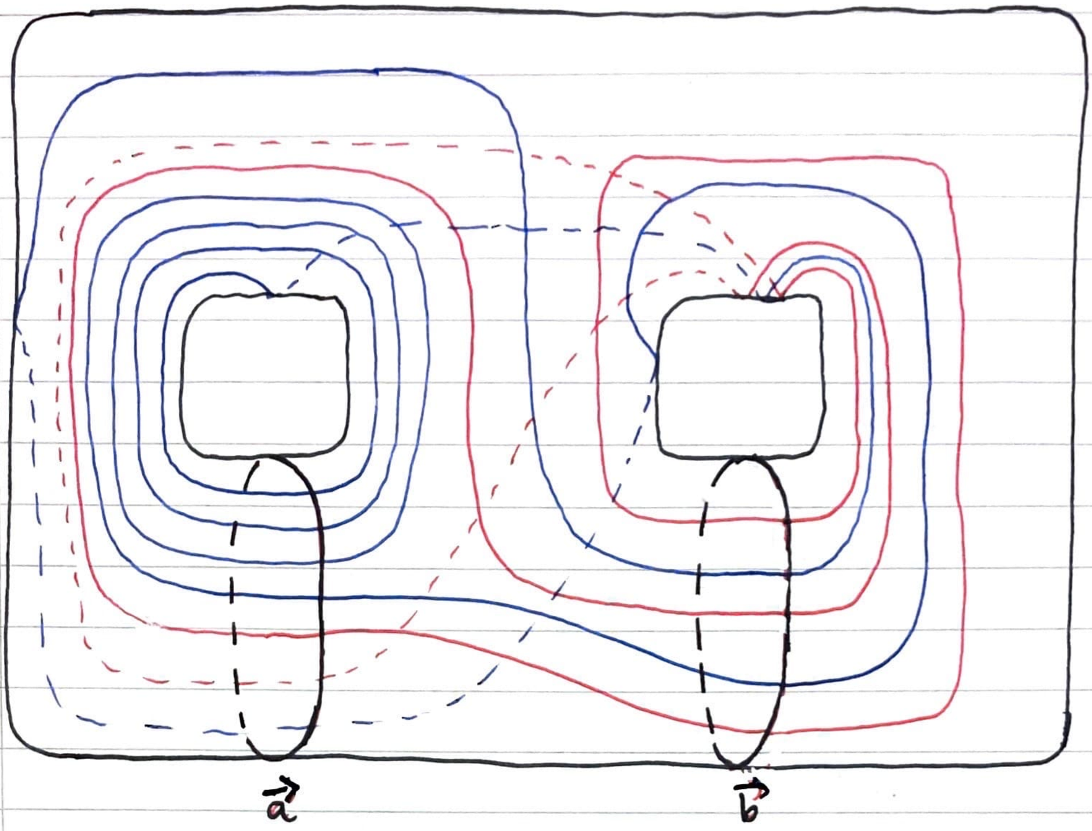


Thm. (Li, Moriah, Pinsky) Berge Conjecture holds for tunnel number one knots.

Q. The $(-2,3,7)$ pretzel knot is a Berge knot. ($^{18}/_1$ and $^{19}/_1$ are lens space slopes.)



Q. Challenge: draw pictures to show K is doubly primitive.



$\langle a, b \mid abbaB, aaaabAb \rangle$

⓪ This is (essentially) the Heegaard diagram in Poincaré's fifth complement (1904) giving the first definition of PHS^3.

Other definitions

① S^3/D^* with $D^* < SU(2)$ the binary dodec. group. (Seifert, Weber, 1933)

② $X_K(1/1)$ where $K = K_{3,2}$ (Dehn, 1910).

③ Take a dodecahedron and glue opposite sides, as in prev. lecture.

④ Some Seifert fibred space over $S^2(a,b,c)$ (Seifert)

Paper: Kirby - Scharleman, "Eight faces of PHS^3."

Q. All of these give PHS^3.