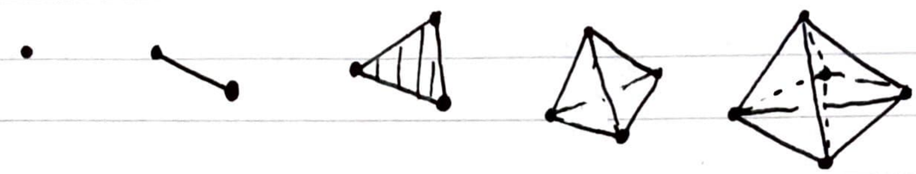
LECTURE 13

TRIANGULATIONS

Defn. The standard n-simplex is
$$\Delta^n = \{x \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum x_i = 1\}.$$

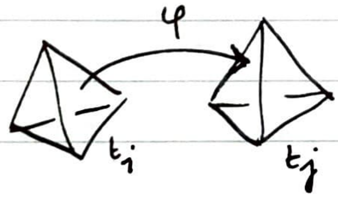
Pictures.



(Δ^{n+1} obtained by coning off Δ^n to a new pt.)

A model tetrahedron is a copy of Δ^3.
A face pairing $\varphi : f_i \to f_j$, $f_i \subset t_i$, $f_j \subset t_j$ is a linear isomorphism.

Picture.

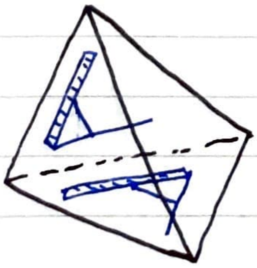


Defn. A triangulation $T = \{(t_i), (\varphi_j)\}$ is a collection of model tetrahedra and face pairings.

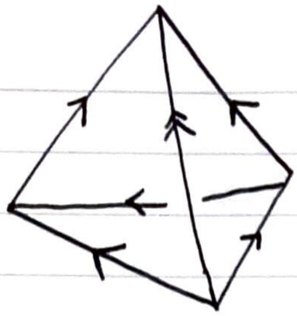
The realization of T is the quotient
$$|T| = \left(\bigsqcup_i t_i\right) / \left(\bigsqcup_j \varphi_j\right).$$

Eg. (1)  $|T| \cong \mathbb{B}^3$

(2)  $|T| \cong \mathbb{B}^3$, but w/ a different cell structure to (1). (glue front faces)

(3)  (glue back faces.)

Q. $|T| = M$ is a manifold.
Find $\chi(M)$, ∂M, $\pi_1(M)$.
Recognise M (find the homeomorphism type).

(4) 

Use same face pairings as in (2) and (3).
Same problem as in (3)!

Q.

HOMEOMORPHISM PROBLEM

Given (M,T), (M',T') triangulated manifolds.
Determine if $M \cong M'$.

Note. This is "easy" in dim ≤ 2.
This is "impossible" in dim ≥ 4.

Thm. (Kuperberg) Geometrisation gives an algorithm to solve the homeo problem in dim 3.

Open. Give upper/lower bounds on the difficulty of the homeomorphism problem in dimension 3.

functional nondeterministic polynomial time

Eg. The function problem of "recognising" spherical space forms (lens spaces are a special case) lies in FNP [Zackenby - Schleimer].
[Also true for $\mathbb{E}^3$, Nil, Solv, $S^2 \times \mathbb{R}$ geometries.]

Suspect: $\mathbb{H}^2 \times \mathbb{R}$, $PSL(2,\mathbb{R})$ geometries are easier to recognise than $\mathbb{H}^3$ geometry.

Thm. (Jackson) Recognising Seifert fibred spaces with boundary lies in FNP.

Difficult cases: SFS over $S^2(a,b,c)$.

Rk. Recognising $\mathbb{H}^3$ manifolds w/ torus boundary seems to be easy in practise.

Eg. (Decision problem, NP)

KNOT GENUS : given a knot $K \subset S^3$, and $g \in \mathbb{N}$, is $g(K) \leq g$.
This lies in NP.
[Hass – Lagarias – Pippenger]

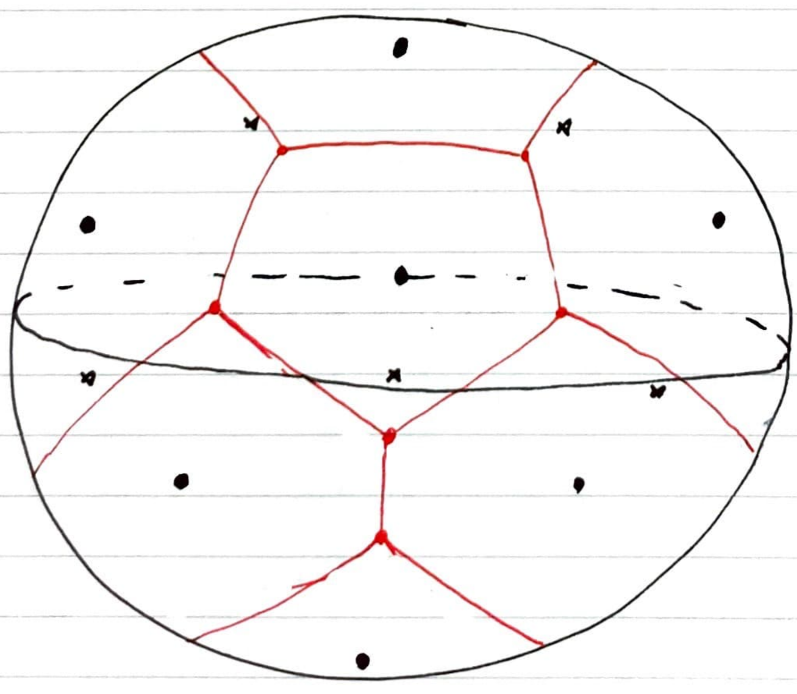
KNOT GENUS lies also in co-NP (can certify the NO answers)
[Lackenby]

FUNCTIONAL KNOT GENUS : given $K \subset S^3$ (as knot diagram), what is $g(K)$?
This lies in FNP [Lackenby].

Recall. D^* is the binary dodecahedral group, that is, the preimage of $D < SO(3)$ (orientation-preserving symmetries of the dodecahedron) in $SU(2)$ under the double cover

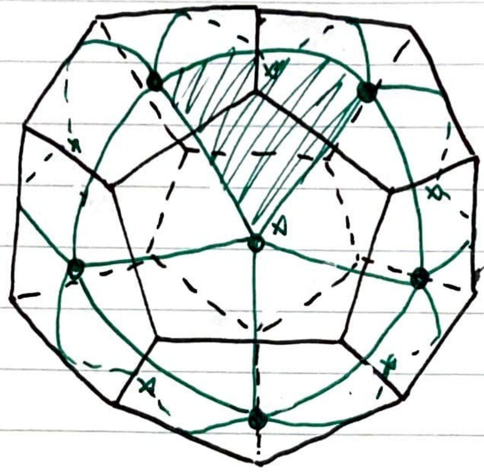
$$SU(2) \xrightarrow{\times 2} SO(3).$$

The 120-cell is the decomposition of S^3 into spherical dodecahedra as Voronoi domains abt. the elt.s of $D^* \subset S^3$.



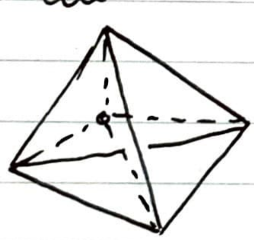
Model: Stereographically project the one-skeleton of the 120-cell to $\mathbb{R}^3$. Cut along the unit two-sphere.

The 600-cell is the dual of the 120-cell.



For every k-cell of the ~~plane~~. primal, the dual has an $(n-k)$-cell.

Eg. 5-cell



16-cell

Eg. Suppose $\zeta_p = \exp\left(\frac{2\pi i}{p}\right) = \zeta$ acts on $S^3 \subset \mathbb{C}^2$ by the usual action, $\zeta \cdot (z,w) = (\zeta\cdot z, \zeta^q \cdot w)$

We gave a triangulation of S^3 with p^2 spherical tetrahedra, invariant under the $\langle \zeta_p \rangle$ action.

This gives a triang. of $L(p,q)$ with p tetrahedra.

Challenge. Pick a generic pt. x of S^3.

Form the orbit $\langle \zeta \rangle . x$ and take Voronoi domains. Dualise to get an invariant triangulation of S^3.

This gives a triangulation of $L(p,q)$.

Conj. This triangulation is minimal (a triang. T of M is minimal if it has the fewest possible tetrahedra).