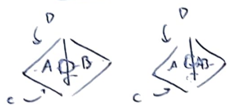
Lecture 14 28-02-2023

Exercise: We gave a triangulation of $L(p,q)$ with p tetrahedra. Find a triangulation of $L(3,1)$ with just two tetrahedra.

Exercise: Find and recognise all 3-manifolds with a one tetrahedron.

Example: The double of a single tet is S^3:



Recall: $\Delta^3 = \{ x \in \mathbb{R}^4 \mid x_i \geq 0, \sum_i x_i = 1 \}$ is the standard tet. Copies of Δ^3 are called model tets. We glue these with face pairings.

A triang $T = \{ (t_i), (\varphi_j) \}$

model — face pairings.

$$|T| = \frac{(\sqcup t_i)}{(\sqcup \varphi_j)}$$ is the realisation.

WANT: $|T|$ is a 3-manifold and $c \subset t$ is an open model cell (vertex, face, edge, tet) then the image of c in $|T|$ is an embedding.

That is: Let $q : \sqcup t_i \to |T|$ be the quotient. Then $q|_c$ is an embedding.

Def: If c is an open cell, then $q^{-1}(q(c))$ is the collection of models of $q(c)$.

Picture:

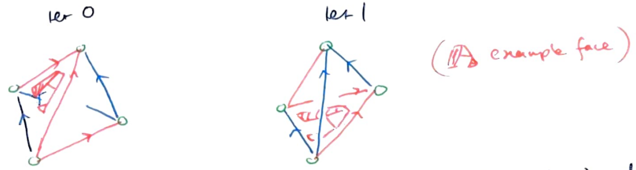
Have four models of e in T.

Lemma: Suppose $|T|$ is closed 3-manifold.

Then:
- tets have 1 model
- faces 2 models
- edges $\geq$ 1 model
- verts $\geq$ 2 models

Ideal Triangulations (Thurston)

Say $T = \{(t_i),(\varphi_j)\}$ is an <u>ideal</u> triang of M if $M \cong |T| - \{\text{vertices}\}$.



Glue <u>front</u> faces of tet_i to <u>back</u> faces of $\text{tet}(i+1) \bmod 2$.

Note $\chi(|T|) = \underset{\text{tet}}{-2} + \underset{\text{faces}}{4} - \underset{\substack{\text{edges}\\ \text{red/blue}}}{2} + 1 = 1$.

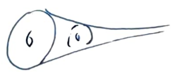
Recall if M closed 3-manifold (or odd), $\chi(M) = 0$.

<u>However</u>:

<u>Exercise</u>: $|T| - \{\text{vertex}\}$ is a non-compact 3-manifold

<u>Exercise</u>: The <u>end</u> of $|T| - \{\text{vert}\} \cong \mathbb{T}^2 \times [0,\infty)$

picture:



To compute the topology of the end, it suffices to understand the topology of the <u>vertex link</u>.

<u>Def</u>: Suppose $v \in |T|$ is a vertex. Let $\{v_i\}$ be the models of v. Fix $\varepsilon << \frac{\sqrt{2}}{2}$. Let $N(v_i, \varepsilon)$ be closed neighbourhoods. Then

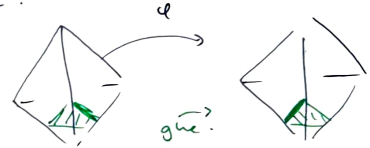
$q(\cup N(v_i,\varepsilon))$ is a vertex <u>star</u>.

Let $S(v_i,\varepsilon)$ be the points at distance ε. Then:

$$lk(v) = q(\cup(S(v_i,\varepsilon)))$$

$$st(v) = q(\cup N(v_i,\varepsilon))$$

<u>Picture</u>:



So, the link of a vertex is a surface equipped with a triangulation.

<u>Exercise</u>: In our example, the vertex link is a 2-torus.
(use Euler characteristic, orientability, for eg).

<u>Remark</u>: Since we use <u>face-pairings</u>, the vertex link is always a surface.

<u>Exercise</u>: Give $T = \{(t_i),(\varphi_j)\}$ so that $|T| - \{\text{vert}\} = M$ is a 3-manifold with Klein bottle cusp. (or with cusp surface being any surface other than a sphere or 2-torus).

Postpone: Geometric triangulations (Thurston).

Angled Triangulations (Rivin, Casson)

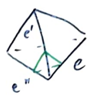
Def: An *angle structure* of T (triang) is a function $\alpha : \{\text{model edges}\} \to [0, \pi]$

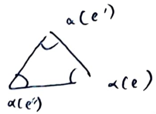
An angle structure is *strict* if the image of α lies in $(0, \pi)$.

We require:

- Vertex: If e, e', e'' are model edges meeting a single model vertex, the
$$\left[\ \alpha(e) + \alpha(e') + \alpha(e'') = \pi.\ \right]$$

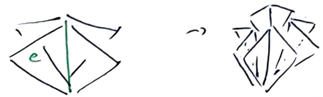
Picture:





cusp triangle is "Euclidean"

- Edge: If $e_0, e_1, \ldots, e_{n-1}$ are the models of an edge $e \in |T|$ then we require
$$\left[\ \sum_i \alpha(e_i) = 2\pi\ \right]$$



Example: In the above (red, blue) example above, we have $\alpha(e) = \frac{2\pi}{6}$ for all models works.

Thm [Rivin, Casson]: If the 'volume functional' $\mathcal{V} : \mathcal{A}(T) \to \mathbb{R}_+$ has a local max in the interior of $\mathcal{A}(T)$ then M is a cusped hyperbolic 3-manifold of finite volume.

Vague plan: If the max occurs in $\partial\mathcal{A}(T)$, we should "retriangulate" and move in the direction of increasing volume (And so prove geometrisation).

Exercise: Compute $\mathcal{A}(T)$ for the example above.

[12 vars, 2 edge eqns, 8 vertex equations (one will be redundant!)]

Question: Find T with $|T| - \{\text{verts}\}$ a cusped 3-manifold. Can $\mathcal{A}(T)$ be empty? A single point?