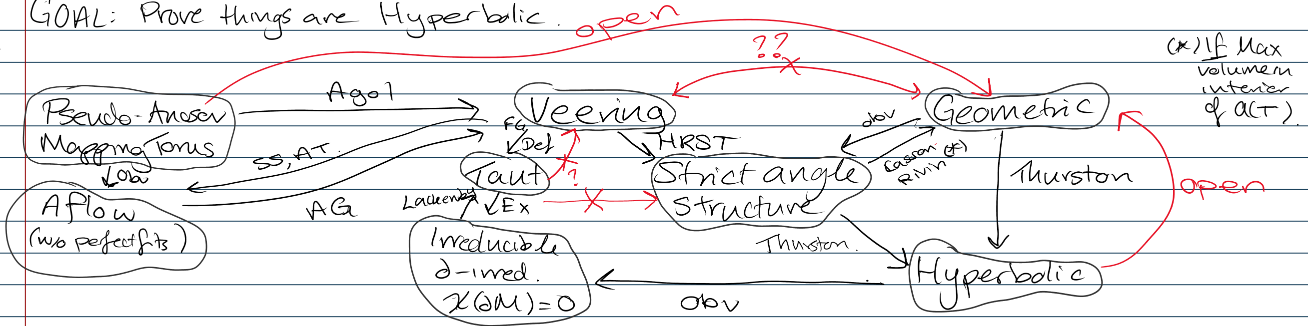
Lecture 15 Geometric Topology

02 March 2023 09:55

GOAL: Prove things are Hyperbolic.



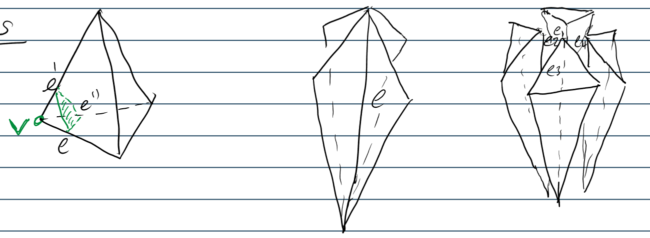
Last time: Suppose T is an ideal triangulation of M. $[\,|T| - \{\text{vertices}\} \cong \text{interior}(M)]$

Suppose $\alpha : \{\text{model edges}\} \to [0,\pi]$ with:

<u>vertex</u> $\alpha(e) + \alpha(e') + \alpha(e'') = \pi$

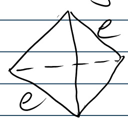
<u>edge</u> $\sum_{g(e_i)=e} \alpha(e_i) = 2\pi$.

Pictures



Exercise If T admits an angle structure, then vertex links have $\chi = 0$ $[\partial M = \sqcup \text{tori},\ \partial M \neq \emptyset]$

Exercise If t is a model tetrahedron and e, e' are model edges of t not sharing a vertex then $\alpha(e) = \alpha(e')$.



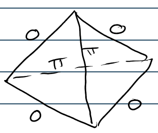
Exercise (!!!) If α is a <u>strict</u> angle structure $(\alpha(e) \neq 0, \pi)$ Then M is hyperbolic.

Recall $\mathcal{A}(T)$ = polytope of angle structures
$\mathcal{A}(T) \subset [0,\pi]^{ME}$ (ME = Model edges).

Taut ideal triangulations [Lackenby]

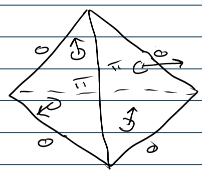
Say α is <u>taut</u> if $\alpha(e) \in \{0, \pi\}$

Picture:



Model Edge of t is <u>equitorial</u> if $\alpha(e) = 0$.

Say (M, T, α) is Transverse Taut if there is a coorientation of the faces which $\begin{pmatrix}\text{agrees}\\ \text{disagrees}\end{pmatrix}$ accross edges of $\alpha = \begin{pmatrix}\pi \\ 0\end{pmatrix}$



Exercise Taut angle structures lie at vertices of $\mathcal{A}(T)$

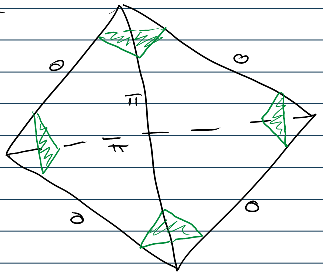
Exercise If (M, T, α) is taut but not transverse taut, then there is a double cover $(\tilde{M}, \tilde{T}, \tilde{\alpha})$ which is transverse taut.

Question Which transverse taut angle structure can be deformed to be strict?

Theorem [Lackenby] Suppose M is compact, $\partial M \neq \emptyset$, $\partial M = \sqcup$ (tori)
M is irreducible, ∂-irreducible
Then for any $[S] \in H_2(M, \mathbb{Z})$ there is a transverse taut triangulation (T, α) carrying a representative $S \in [S]$.

Ex Find transverse taut triangulation on $S_{g,1} \times S^1$ $g \geq 1$.

Picture



So $lk(v)$ = link of v is made of bigons if (T, α) is taut.

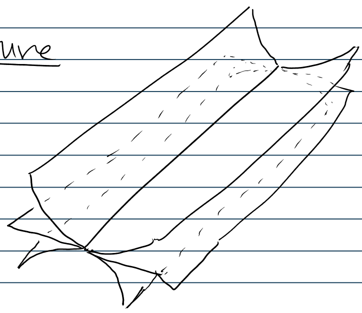
Veering Triangulations [Agol]

Suppose (M, T, α) is transverse (oriented) taut.
Say (M, T, α) is veering if there is a colouring of the edges of $|T|$ by red, blue, pulling back to the models like so:

Red = Right

Blue = Left.

Picture

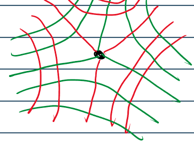


Neighbourhood of an edge in a taut triangulation.

Theorem [Gueritaud, Futer, RHST]
If (M, T, α) is veering then α deforms to be strict.

Theorem [Agol] Suppose $f: S \to S$ is pseudo-Anosov homeo of a surface.

Let $S^\circ = S - \{\text{singularities}\}$
Let $f^\circ = f|S^\circ$
Then the Mapping Torus $\left(M_{f^\circ} = \dfrac{S^\circ \times [0,1]}{(x,1) \sim (f^\circ(x), 0)}\right)$
admits a canonical veering triangulation



Theorem [Agol] Suppose $B > 0$. Then the collection of veering triangulations, admits a canonical veering triangulation

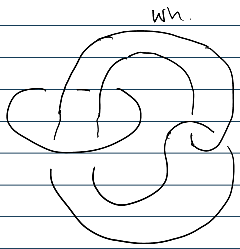
Theorem [Agol] Suppose $B > 0$. Then the collection of veering triangulations, arising from drilled mapping tori M_{f° where
$$|\chi(S^\circ)| \log(\lambda(f^\circ)) \leq B,$$ is finite

Point the map

$\{\text{PA homeo}\} \longrightarrow \{\text{veering triangulations}\}$
is often infinite-to-one

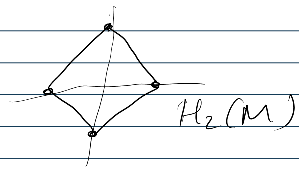
Theorem [S-Segerman, Agol-Tsang]
If $(M, \mathcal{T}, \alpha)$ is veering, then there is a canonically associated Pseudo-Anosov Flow. (w/o perfect fits)

Theorem [Thurston]
Suppose M is atoroidal irreducible, ∂irreducible, acylindrical (conn, orient)
Then there is a polyhedral norm $|\cdot| : H_2(M, \mathbb{R}) \to \mathbb{R}$ so that if some interior point of a face of the unit ball is fibered, then all interior points of that face are fibered.



$M = S^3 - Wh$

$H_2(M, \mathbb{R}) \cong \mathbb{R}^2$



Theorem [SS] $M = S^3 - Wh$ admits no veering triangulation.