3/3/23 GEOMETRIC TOPOLOGY, SAUL SCHLEIMER

LECTURE 16

HYPERBOLIC GEOMETRY

Two models for $\mathbb{H}^3$:

$\underline{UHS} = \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}$.

$\underline{BM} = \{x \in \mathbb{R}^3 \mid |x| < 1\}$.

$$ds_{UHS} = \frac{ds_E}{t} \quad \text{and} \quad ds_{BM} = \frac{2 ds_E}{1-r^2}, \; r^2 = x_1^2 + x_2^2 + x_3^2.$$

Fix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$.

Set $q = z + tj$ in the quaternions (i.e. $q = x + iy + tj$).

Define. $\gamma(q) = (aq+b)(cq+d)^{-1} = (cq+d)^{-1}(aq+b)$

Q. Check this equality (the quaternions are not commutative) and show that $\gamma(q) \in UHS$, i.e. k-coord. is zero, j-coord. > 0.

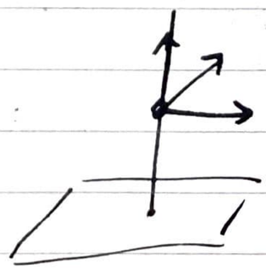
Q. γ acts via isometry on UHS. [Ahlfors]

Q. The action factors through $PSL(2,\mathbb{C})$.

Q. The stabiliser of $(0,0,1)$ in $SL(2,\mathbb{C})$ is $SU(2)$.

Thus, $PSL(2,\mathbb{C})$ acts simply transitively on right-handed orthonormal frames to UHS.

Thus, $SL(2,\mathbb{C}) \cong SU(2) \times \mathbb{H}^3$.

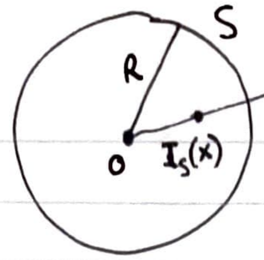


std. frame

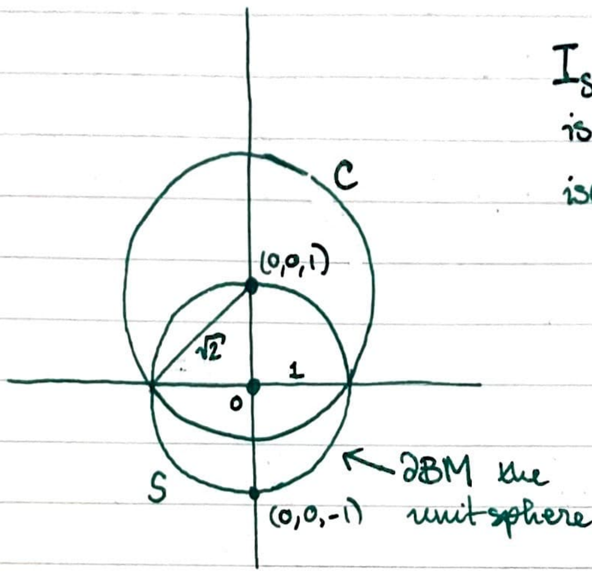
(Simply transitively means that γ determines and is determined by its action on the std. frame.)

Thus, $PSL(2,\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$.

Q. Every $\gamma \in \mathrm{Isom}^+(\mathbb{H}^3)$ is a product of two inversions in spheres.



Inversion in the sphere S:
$I_S : \hat{\mathbb{R}}^3 \to \hat{\mathbb{R}}^3$
preserves rays, and
$|I_S(x)||x| = R^2$.

L. 

$I_S \circ I_C$: UHS $\to$ BM is an (orientation-preserving) isometry.

<u>So</u> can conjugate action of $PSL(2,\mathbb{C})$ by $I_S \circ I_C$ and realize $\mathrm{Isom}^+(BM)$ as a subgroup of (orientation-preserving) Möbius transformations (where a Möbius transf. is any product of inversions in spheres).

Rk. Note that UHS, BM are conformal models, i.e. angles in the model and in $\mathbb{E}^3$ agree.

Pf. [$ds_{UHS} = \dfrac{ds_{\mathbb{E}}}{r}$.

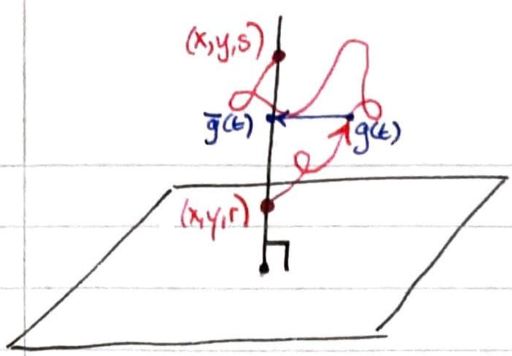
Not conformal: Klein model, hyperboloid model, exponential model.

GEODESICS (following P. Scott's BLMS article)

[BTW. Skipped: types of isometry (identity, elliptic, parabolic, hyperbolic, loxodromic).

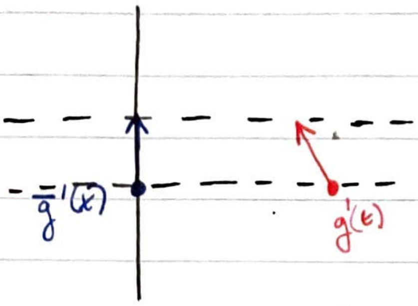
L. Vertical arcs in UHS are geodesics.

Pf. Suppose $g : [0,1] \to$ UHS has
$g(0) = (x,y,r)$, $g(1) = (x,y,s)$.



Set $g(t) = (g_1(t), g_2(t), g_3(t))$.

Define $\bar{g}(t) = (x, y, g_3(t))$.

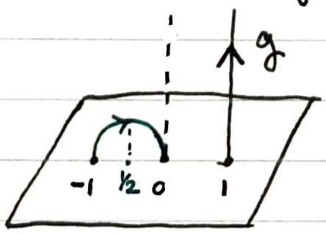


<u>Claim</u>: $|g'(t)| \geq |\bar{g}'(t)|$, with inequality if g is not vertical.

$\Rightarrow$ length$(g) \geq$ length$(\bar{g})$.

Q. If g_3' changes sign, then $\bar{g}$ is not a geodesic.
(We could chop out bits of the path and get a shorter path)

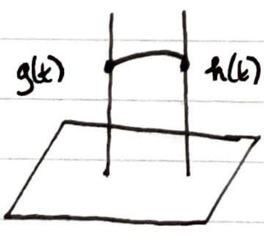
Z. The element $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ sends $g(t) = 1 + tj$ to a hemi-circle from -1 to 0.



Cor. Arcs of hemicircles are geodesics.

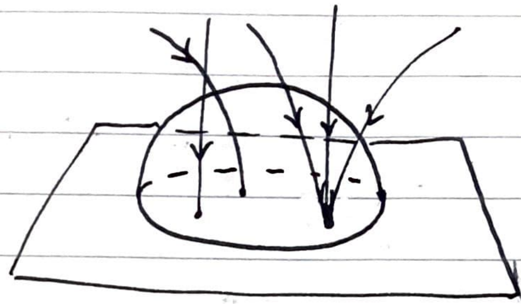
Q In fact, $d_{UHS}((0,0,r),(0,0,s)) = \log(s/r)$ (if $s > r$).

Q. Suppose g, h are vertical geodesics, parametrized with unit speed
Show that the distance $d_{\mathbb{H}^3}(g(t), h(t)) =$ const. + exp. decay.



Defn. Call two such geodesics asymptotic.
This is an equivalence relation, on oriented geodesics.
Take quotient to get $\partial_\infty \mathbb{H}^3$.

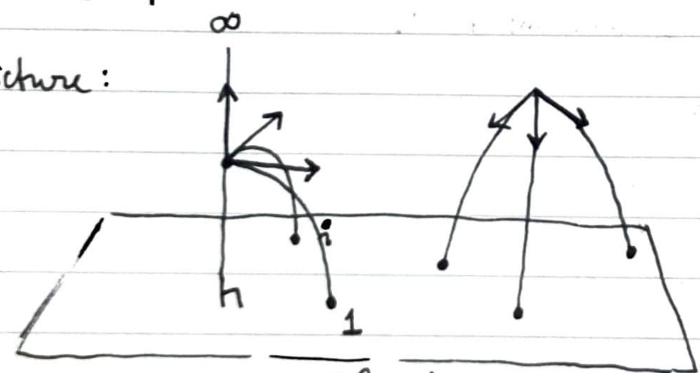
get a topology from hemispheres.
If H is a hemisphere (transversely-oriented) perpendicular to $\mathbb{C}$, then all geodesics crossing H in same direction is an open set.



Q. $\partial_\infty \mathbb{H}^3 \cong S^2$.

Key fact: the action of $PSL(2,\mathbb{C}) \cong \mathrm{Isom}(\mathbb{H}^3)$ on $\partial_\infty \mathbb{H}^3$ is simply three-transitive.
[γ determines and is determined by its action on any three distinct points of $\partial_\infty \mathbb{H}^3 \cong \mathbb{C} \cup \{1/0\}$.]
[Hint for proof: use KAN decomposition of $\gamma \in PSL(2,\mathbb{C})$.

[Picture:



frames $\sim$ triples of points on the boundary

By applying a ① parabolic matrix, we can move the base pt. to the ~~basept. of the std~~ line above 0,
by applying a ② hyperbolic matrix, we can slide the base pt. along the line to the basept. of the std. frame,
by applying an ③ elliptic matrix, we can rotate the frame to the std. frame!]

Claim. The frame bundle over $\mathbb{H}^3$ is isomorphic to $(S^2)^3 \setminus \Delta$, i.e. $(\partial_\infty \mathbb{H}^3)^3 \setminus \Delta$.