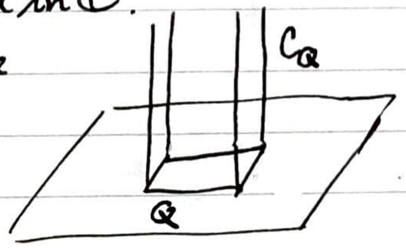
14/3/23 GEOMETRIC TOPOLOGY, SAUL SCHLEIMER

LECTURE 20

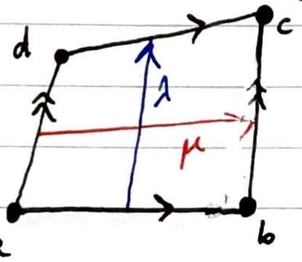
Owed. Proof that $(S^3 - \text{fig. } 8, T, z)$ is complete for $z(e) \equiv \omega$.

CHIMNEYS

Suppose Q is a quadrilateral in $\mathbb{C}$.
Let $C_Q = Q \times \mathbb{R}_{>0}$. Picture



Let S_Q be the similarity surface defined by



glue $\begin{cases} [a,d] \text{ to } [b,c] \\ [a,b] \text{ to } [d,c] \end{cases}$

via unique similarities.

So. $\pi_1(S_Q) = \langle \mu, \lambda \rangle \cong \mathbb{Z}^2$.

Let $\rho_Q : \pi_1(S_Q) \longrightarrow PSL(2,\mathbb{C})$ be the homomorphism where

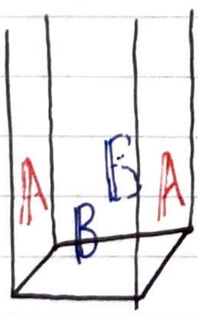
$$\rho_Q(\mu) : \begin{matrix} \infty \longmapsto \infty \\ a \longmapsto b \\ d \longmapsto c \end{matrix} \quad \Bigg| \quad \rho_Q(\lambda) : \begin{matrix} \infty \longmapsto \infty \\ a \longmapsto d \\ b \longmapsto c \end{matrix}$$

Q. ρ_Q is well-defined.
[I.e. $\rho_Q(\mu)$, $\rho_Q(\lambda)$ commute and have no other relations.]

Set $\Gamma_Q = \operatorname{im}(\rho_Q) \leq PSL(2,\mathbb{C})$.
Set $M_Q = \mathbb{H}^3 / \Gamma_Q$.

Note. C_Q is a fundamental domain for M_Q.

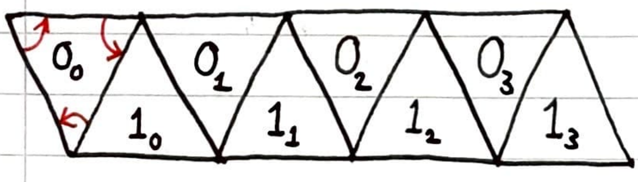
Picture



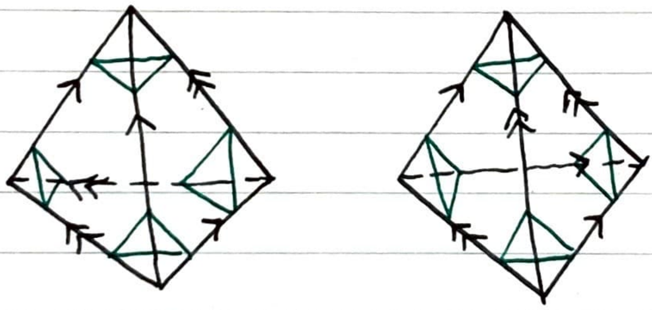
Q. $M_Q \cong T \times (0,\infty)$.

Prop. M_Q is hyperbolic and infinite volume.
M_Q is complete iff S_Q is a translation surface.

Note. Thus $(M = S^3 \setminus \text{fig. } 8, T, z)$ is complete, because the cusp torus is a translation surface.

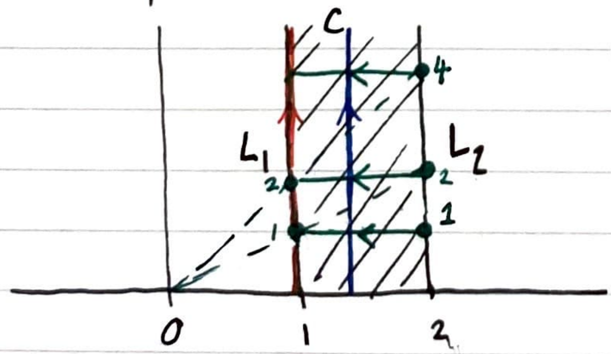


All complex angles $= \omega$
$= \frac{1}{2} + i\frac{\sqrt{3}}{2}$.



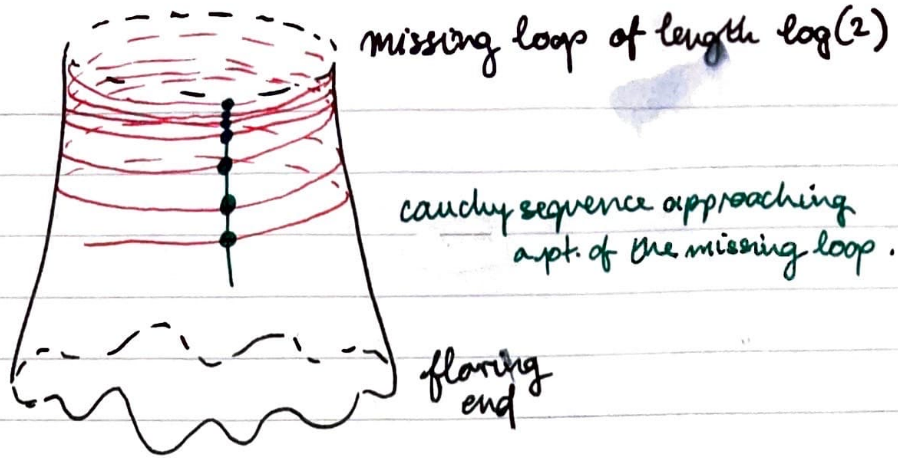
So. Prop. $\Rightarrow$ Thm. [from last time]

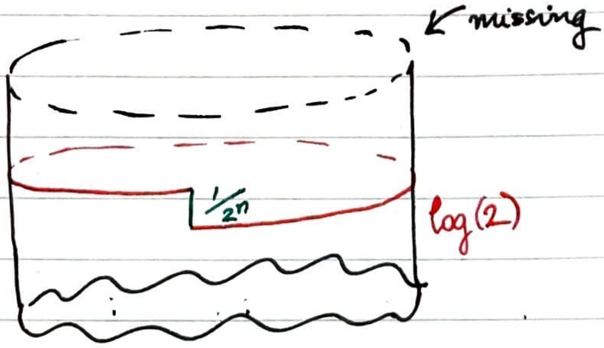
Eg. Non-completeness in dimension 2



Glue L_2 to L_1 by $z \mapsto z/2$.
This metric space is not complete, $C/z \mapsto z/2$

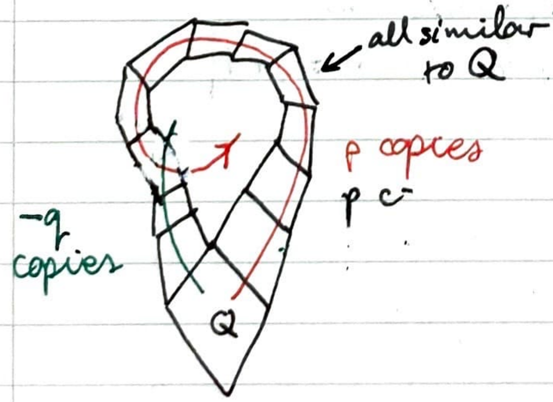
Cartoon.



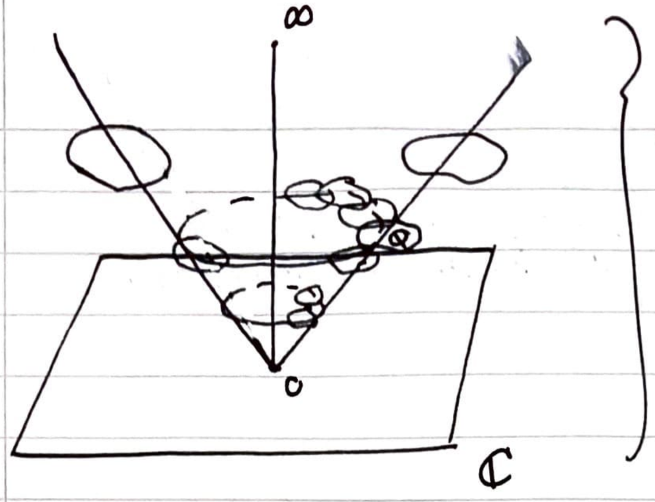


Hence [via geom. limit argument] the "incompleteness locus" is a geodesic, which can be determined.

Eg. Non-completeness in dimension 3.
Looking from above:



View from side:
Q Suppose S_Q is not a translation surface.
Then $\rho_Q(\mu)$, $\rho_Q(\lambda)$ have a common invariant axis (vertical).



Suppose the axis is
$0 \times (0,\infty)$.
It is the incompleteness locus

Star Wars analogy (watch the Phantom Menace for context).

In $\mathrm{Dev}: \widetilde{M}_Q \longrightarrow \mathbb{H}^3$ [conjugating correctly],
if Q is not a parallelogram, then $\mathrm{Im}(\mathrm{Dev})$ misses exactly the axis $0 \times (0,\infty)$.

Prop. Suppose $h(\mu)$, $h(\lambda)$ are the homothety coefficients of $\rho_Q(\mu)$, $\rho_Q(\lambda)$.

The metric completion of M_Q is a hyperbolic three-manifold
<u>iff</u> (i) Q is a parallelogram

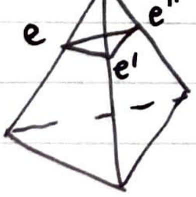
<u>or</u>

(ii) there are coprime $p, q \in \mathbb{Z}$ with
$$\boxed{p \log(h(\mu)) + q \log(h(\lambda)) = 2\pi i}$$

GEOMETRIC TRIANGULATIONS

Suppose (M, T, z) is a manifold M with an ideal triangulation T $\left[\,|T| - \{\text{vertices}\} \cong \mathring{M}\,\right]$
and edge parameters $\{z(e) \mid e \text{ model edge}\}$.

We call (T, z) a geometric triangulation if
(0) tetrahedra: for e, e', e'' model edges sharing a model vertex [anticlockwise],
$$z(e') = \frac{-1}{z(e)-1}, \quad z(e'') = \frac{-1}{z(e')-1}$$



(1) Positivity: $\mathrm{im}(z(e)) > 0$ for all model edges.

(2) Edges: if e is an edge,

$$\sum_{[e]=e} \log(z(e_i)) = 2\pi i$$

(Note. $\prod z(e_i) = 1$ does not suffice.)

(3) S_Q is a translation surface, where Q is any cusp, and S_Q is the corresponding cusp torus.

Thm. If (M, T, z) is a geometric triangulation, then $M \cong |T| - \{\text{vertices}\}$ is hyperbolic finite volume and complete.

Question

Prop. If (M, T, z) is geometric, then $(M, T, \arg(z))$ is an angle structure.