Please let me know if any of the problems are unclear or have typos. Please let me know if you have suggestions for exercises. Each problem is followed by a number in brackets – this is the number of marks the question is worth. Please turn in at least 10 marks worth of answers by the end of week three. You should feel free to work with others; if you do so, please say whom you collaborate with (and acknowledge other sources as necessary).

Exercise 1.1. Prove that the dimension of a manifold is well-defined. [2]

Exercise 1.2. Prove that the n–sphere S^n is an n–manifold. [2]

Exercise 1.3. For each hypothesis of manifoldness, find a topological space that satisfies the other two hypotheses, but not the given one. [3]

Exercise 1.4. Suppose that M and N are m– and n–manifolds respecively. Prove that the product space $M \times N$ is an $(m+n)$–manifold. [2]

Exercise 1.5. Determine all homeomorphisms between spheres and tori. [2]

Exercise 1.6. Recall that the formal definition of a torus is $\mathbb{T}^0 = \{\mathrm{pt}\}$ and $\mathbb{T}^{n+1} = \mathbb{T}^n \times S^1$. Show that $\mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n$. [3]

Exercise 1.7. Show that the S^n is homeomorphic to the disjoint union of a pair of n–balls, glued along their boundaries. [2]

Exercise 1.8. Show that the S^3 is homeomorphic to the disjoint union of a pair of solid tori ($D^2 \times S^1$), glued along their boundaries. [2]

Exercise 1.9. Show that $\mathbb{RP}^n$ is an n–manifold. Show that $\mathbb{CP}^n$ is a $2n$–manifold. Show that $\mathbb{RP}^1 \cong S^1$. Show that $\mathbb{CP}^1 \cong S^2$. [4]

Exercise 1.10. Suppose that B^n is the n–ball.

a) Prove that B^n is an n–manifold with boundary (namely, S^{n-1}).

b) Prove that B^n is not an n–manifold. [2]

Exercise 1.11. Suppose that $M = M^n$ is an n–manifold with boundary. Suppose that ∂M is non-empty.

a) Prove that ∂M is an $(n-1)$–manifold.

b) Prove that $\partial(\partial M)$ is empty. [3]

Exercise 1.12. We use the notation $I = [0, 1]$. Here are three definitions of the Möbius band:

a) $M = M^2 = \mathbb{R} \times I/(x, y) \sim (x + 1, 1 - y)$

b) $M = I^2/(0, y) \sim (1, 1 - y)$

c) M is the "half-twisted band" in $\mathbb{R}^3$

Prove that these are all homeomorphic. [4]

Exercise 1.13. With M being the Möbius band, show that $\mathbb{RP}^2 \cong M^2 \cup_\partial D^2$. [4]

Exercise 1.14. Give an elegant drawing of $\mathbb{RP}^2$. [4]

Exercise 1.15. Prove that ambient isotopy is an equivalence relation on (tame) knots in the three-sphere. [2]

Exercise 1.16. The first four knots in the Rolfsen knot table are 0_1 (the unknot), 3_1 (the trefoil), 4_1 (the figure-eight), and 5_1 (the cinquefoil). Draw pictures of these. Prove that no two of them are ambiently isotopic in the three-sphere. [5]

Exercise 1.17. Suppose that K is a tame knot in the three-sphere. Suppose that U_K is an closed regular neighbourhood of K. Let $X_K = S^3 - \text{interior}(U_K)$. Prove that $S^3 - K$ deformation retracts to X_K (so in particular is homotopy equivalent). [2]

Exercise 1.18. Show that the Gordon–Luecke theorem (a knot in the three-sphere is determined by its complement) does not hold for links. [4]