Please let me know if any of the problems are unclear or have typos. Please let me know if you have suggestions for exercises. Each problem is followed by a number in brackets – this is the number of marks the question is worth. Please turn in at least 10 marks worth of answers. You should feel free to work with others; if you do so, please say whom you collaborate with (and acknowledge other sources as necessary).

Exercise 3.1. Suppose that p and q are positive integers which are greater than one and which are coprime. Determine which pairs of the eight oriented torus knots, with coefficients $(\pm p, \pm q)$ and $(\pm q, \pm p)$, are isotopic. [3]

Exercise 3.2. Suppose that p and q are as in the previous problem. Consider the action of S^1 on S^3 where

$$e^{i\theta} \cdot (z, w) = (e^{qi\theta} z, e^{pi\theta} w)$$

Show that the critical orbits (that is, those not isotopic to nearby orbits) of the action are exactly the circles C_z and C_w: the intersection of S^3 with the planes $w = 0$ and $z = 0$. Show that the generic orbits of the action are isotopic to the (p, q)–torus knot. [3]

Exercise 3.3. Suppose that p and q, and the action of S^1 on S^3, are as in the previous problem. Show that the quotient S^3/S^1 is a two-sphere. Show that the quotient has a natural spherical metric away from the points $[C_z]$ and $[C_w]$. Show that these points are cone points with angles $2\pi/q$ and $2\pi/p$ respectively. [4]

Exercise 3.4. Make explicit the isomorphism $S^3 \cong \mathrm{SU}(2)$. We now use the action of Exercise 3.2, but with $p = q = 1$. Express this action as that of a subgroup acting on SU(2) (say, by right multiplication). Deduce that the orbits of the action are cosets of this subgroup. [3]

Exercise 3.5. Let UQ be the unit quaternions. Let $\widetilde{\mathrm{SO}}(3)$ be the universal cover of SO(3). Prove that these are both isomorphic to SU(2) (as Lie groups). [3]

Exercise 3.6. Suppose that K and K' are knots. Prove that the connect sum $K \# K'$ is a satellite knot. [2]

Exercise 3.7. Suppose that M is a manifold equipped with a Riemannian metric. We define $\mathrm{UT}(M)$ to be its *unit tangent bundle*. Prove the following pairs of spaces are homeomorphic.

- $\mathrm{UT}(S^1) \cong S^0 \times S^1$
- $\mathrm{UT}(S^2) \cong \mathrm{SO}(3)$
- $\mathrm{UT}(T^2) \cong S^1 \times T^2$
- $\mathrm{UT}(S^3) \cong S^2 \times S^3$

Using the above, or otherwise, prove that $\mathrm{UT}(S^2)$ is not homeomorphic to $S^1 \times S^2$. [3]

Exercise 3.8. Suppose that $M = \mathbb{R}^2 - \{(0,0)\}$. Suppose that A is the diagonal matrix with non-zero entries 2 and 1/2, in that order. We define an action $\rho\colon \mathbb{Z} \times M \to M$ by taking $\rho(n,(x,y)) = A^n(x,y)$.

- Prove that ρ is smooth and free.
- Prove that ρ is not properly discontinuous.
- Prove that M/ρ is not Hausdorff.

More generally, describe the quotient M/ρ. [5]

Exercise 3.9. Suppose that D is a regular dodecahedron. Form P, the *dodecahedral space*, by identifying opposite faces of D with a one-tenth right-handed twist.

- Show that P is an oriented three-manifold.
- Give a presentation of $\pi_1(P)$.

Using this, or otherwise, prove that P is a integral homology three-sphere. (That is, for all k we have $H_k(P,\mathbb{Z}) \cong H_k(S^3,\mathbb{Z})$.) [3]

Exercise 3.10. Suppose that D is a regular dodecahedron. Let $\Gamma < \mathrm{SO}(3)$ be the group of orientation-preserving symmetries of D. Let $\Gamma*$ be the preimage of Γ in $\widetilde{\mathrm{SO}}(3)$. This is called the *binary dodecahedral group*. Using the identification of $\widetilde{\mathrm{SO}}(3)$, show that the Voronoi cells about $\Gamma*$ gives the 120–cell. [3]

Exercise 3.11. Prove that $\widetilde{\mathrm{SO}}(3)/\Gamma*$ (as given in Exercise 3.10) is homeomorphic to P, (as given in Exercise 3.9). [2]

Exercise 3.12. Suppose that p, q, p', and q' are non-zero integers with $\gcd(p,q) = \gcd(p',q') = 1$. Prove the following: if $p' = p$ and $q' = \pm q^{\pm 1} \pmod{p}$ then $L(p,q)$ is homeomorphic to $L(p',q')$. [3]

Exercise 3.13. Suppose that $M = L(p,q)$ is a lens space. Show that the Clifford torus T in S^3 descends to give a torus T' in M which bounds solid tori U' and V' on both sides. Suppose that D' and E' are meridian disks for U' and V', respectively. Describe (up to isotopy) how $\partial D'$ and $\partial E'$ lie in T'. [2]

Exercise 3.14. Prove the following homeomorphisms.

- $L(1,1) \cong S^3 \cong \mathrm{SU}(2) \cong \widetilde{\mathrm{SO}}(3) \cong \mathrm{UQ}$
- $L(2,1) \cong \mathbb{RP}^3 \cong \mathrm{PSU}(2) \cong \mathrm{SO}(3) \cong \mathrm{UT}(S^2)$
- $L(4,1) \cong \mathrm{UT}(\mathbb{RP}^2)$.

You may (and should) freely use the results claimed in Exercises 3.5 and 3.7. [4]