Please let me know if any of the problems are unclear or have typos. Please let me know if you have suggestions for exercises. Each problem is followed by a number in brackets – this is the number of marks the question is worth. Please turn in at least 10 marks worth of answers. You should feel free to work with others; if you do so, please say whom you collaborate with (and acknowledge other sources as necessary).

Exercise 4.1. Suppose that M is a compact connected oriented three-manifold (with boundary). Suppose that M has a handle structure with one 0–cell, two 1–cells, and one 2–cell. List all possibilities for ∂M. [2]

Exercise 4.2. Suppose that M is a compact connected oriented three-manifold, with boundary a two-torus T. Suppose that α is an essential simple closed curve in T; we call α a *slope* in T. Let $A = N(\alpha)$ be an annulus neighbourhood of α in T. Suppose that $V \cong D^2 \times I^1$ is a two-handle (that is, disk cross interval). Let $B = \partial D^2 \times I^1$ be the attaching region for V. Suppose that $\phi, \psi \colon B \to A$ are homeomorphisms. We form $M_\phi = M \sqcup V/\phi$ and $M_\psi = M \sqcup V/\psi$. Show that ∂M_ϕ is a two-sphere. Show that M_ϕ is homeomorphic to M_ψ [5]

Exercise 4.3. Suppose that M is a compact connected oriented three-manifold, with boundary a two-sphere. Suppose that $U \cong B^3$ is a three-ball. Suppose that $\phi, \psi \colon \partial M \to \partial B^3$. We form $M_\phi = M \sqcup U/\phi$ and $M_\psi = M \sqcup U/\psi$. Show that these manifolds are homeomorphic. [3]

Exercise 4.4. Suppose that M is a compact connected oriented three-manifold, with boundary a two-torus. Suppose that $\alpha \subset \partial M$ is a slope. Show that the homeomorphism type of the *Dehn filling* $M(\alpha)$ is well-defined. (That is, is independent of the various choices made when gluing a solid torus $V = S^1 \times D^2$ to M so that the boundary of a meridian disk glues to α.) [2]

Exercise 4.5. Suppose that M is a solid torus. List all homeomorphism classes of three-manifolds obtained by Dehn filling M. [2]

Exercise 4.6. Suppose that M is a compact connected oriented three-manifold. Let $\iota \colon \partial M \to M$ be the resulting inclusion. Let $\iota_* \colon H_1(\partial M, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ be the induced homomorphism. Prove that

$$\mathrm{rk}(\ker(\iota_*)) = \frac{1}{2}\,\mathrm{rk}(H_1(\partial M, \mathbb{Z}))$$

That is, the torsion-free rank of the kernel is have that of the first homology of the boundary. [3]

Exercise 4.7. Suppose that M is a compact connected oriented three-manifold with boundary a two-torus. Prove that there is a unique slope λ in ∂M that bounds a connected oriented properly embedded surface $(F, \partial F) \subset (M, \partial M)$. [3]

Exercise 4.8. Suppose that M is a compact connected oriented three-manifold with boundary a two-torus. Prove that there is a unique slope λ in ∂M that bounds a connected oriented properly embedded surface $(F, \partial F) \subset (M, \partial M)$. [3]

Exercise 4.9. Suppose that K is a smooth knot in S^3. Suppose that F is a smooth Seifert surface for K. Since K is the boundary of F, the tangent plane to F at points of K decomposes as a sum of two lines – one tangent to K and the other normal to K.

- Prove that the normal line field is not twisted; in particular it gives a normal vector field X.
- Prove that the knots $K + \epsilon X$ and $K - \epsilon X$ are isotopic in the complement of K. [3]

Exercise 4.10. Prove one direction of the disk theorem. [2]

Exercise 4.11. Show that the two-sided hypothesis, in the disk theorem, is necessary. [2]

Exercise 4.12. Suppose that F is a compact connected surface, without boundary, embedded in the three-sphere. Prove that the following.

- F is two-sided.
- F is orientable.
- F is a two-sphere, or F is compressible. [5]

Exercise 4.13. Suppose that K is a tame knot in S^3. Suppose that V is a closed regular neighbourhood of K in S^3, and $U \subset V$ is its interior. Recall that $X_K = S^3 - U$ is the associated *knot exterior*. Prove the following.

- X_K is an integral homology $S^1 \times D^2$.
- $X_K(1/0)$ is homeomorphic to S^3.
- $X_K(1/n)$ is an integral homology S^3, for all $n \in \mathbb{Z}$.
- $X_K(0/1)$ is an integral homology $S^1 \times S^2$. [8]

Exercise 4.14. Give an example of a tame knot $K \subset S^3$ which has a pair of non-isotopic Seifert surfaces of minimal genus. [5]

Exercise 4.15. Suppose that K is the (p, q)–torus knot. Find a Seifert surface for K with genus $(p - 1)(q - 1)/2$.

Exercise 4.16. Suppose that T is a standard (Clifford) torus embedded in S^3. Suppose that K is the (p, q)–torus knot lying on T. Let $N(k)$ be a closed regular neighbourhood of K; let $n(K)$ be its interior. Thus $A = T - n(K)$ is an annulus.

Suppose that F is the Seifert surface you found in Exercise **??**. The intersection $A \cap \partial N(K)$ gives the *annulus slope* α in $\partial N(K)$. The intersection $F \cap \partial N(K)$ gives the *longitudinal slope* λ in $\partial N(K)$. As usual we use μ to denote the *meridional slope* in $\partial N(K)$.

Describe how F meets T. Using this (or otherwise) compute the algebraic intersection number of α with μ and with λ. [4]