INTRODUCTION TO TOPOLOGY

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Lecture 1

> *"[...] so far as geometry is concerned, we need still another analysis which is distinctly geometrical or linear and which will express situation [situs] directly as algebra expresses magnitude directly. "*
> — G.W.Leibniz, Letter to Huygens, Sept. 8, 1679

Topology is the study of properties of sets that are invariant under continuous deformations; it is concerned with concepts such as "nearness", "neighbourhood", and "convergence". An often cited example is that a cup is topologically equivalent to a torus, but not to a sphere. But what exactly does "topologically equivalent" mean?

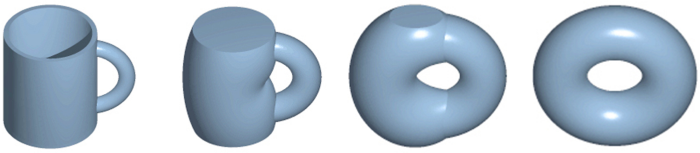

Figure 1.1: A cup morphing into a torus. (c) LucasVB (Wikipedia)

The roots of topology go back to the work of Leibniz and Euler in the 17th and 18th century. It was only towards the end of the 19th century, through the work of Poincaré, that topology began taking shape as a subject in its own right. His seminal paper "Analysis Situs" from 1895 introduced, among other things, the idea of a homeomorphism and the fundamental group. Nowadays, topological ideas are an indispensable part of many fields of mathematics, ranging from number theory to partial differential equations.

1.1 Background and terminology

This course assumes familiarity with metric spaces, linear algebra, some algebra (group theory), and calculus. We use capital letters X, Y, Z to denote sets and A, B, C to denote subsets. We often use U and V to label open sets and write $I = [0, 1]$ for the unit interval in $\mathbb{R}$. The notation $A \subset X$ denotes (not necessarily proper) inclusion, and $X - A$ is the complement of A in X. Following common pedantry, we will refer to "sets" of sets as collections of sets, to avoid logical catastrophes (the "set of sets that are not members of themselves"). Other notation will be explained as it arises. Following common convention, we use arrows and diagrams to describe maps between sets. A diagram such as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

describes three sets X, Y, Z and three functions, $f\colon X \to Y$, $g\colon Y \to Z$, and $h\colon X \to Z$. Such a diagram is *commutative* if all compositions agree; here, this means that $h = g \circ f$. We sometimes use the notation $X \hookrightarrow Y$ to denote an *injective* (or one-to-one) map (for example, the map $x \mapsto x$ arising from an inclusion $X \subset Y$), and $X \twoheadrightarrow Y$ for a *surjective* (or onto) map. We recall the definition of a metric space.

Definition 1.1. A *metric space* is a set X, together with a function $d\colon X \times X \to \mathbb{R}$, such that for all $x, y, x \in X$,

1. (positivity) $d(x, y) \geq 0$, with equality if and only if $x = y$;
2. (symmetry) $d(x, y) = d(y, x)$;
3. (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

Well known examples include $\mathbb{R}^n$ with the Euclidean distance

$$d(x, y) = \|x - y\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

any distance induced by a norm, or the space $C([0, 1])$ of real-valued, continuous functions on the interval $[0, 1]$, with the metric

$$d(f, g) = \int_0^1 |f(x) - g(x)| \, \mathrm{d}x.$$

Given a metric space (X, d) and $x_0 \in X$, we denote by

$$B(x_0, \varepsilon) = \{x \in X \colon d(x, x_0) < \varepsilon\}$$

the open ball of radius ε centred on x_0.

Definition 1.2. Let (X, d) be a metric space. A set $U \subset X$ is called *open* in X, if for every $x \in U$ there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. A subset of X is *closed* in X if its complement is open.

Clearly, if (X, d) is a metric space, then the empty set $\emptyset$ and the whole set X are open. Moreover, the union of *any* collection of open sets is open, and the intersection of a *finite* collection of open sets is open (show this!). It turns out that these properties allow us to define open sets and neighbourhoods beyond metric spaces.

1.2 Topological spaces

Definition 1.3. A **topological space** is a set X, together with a collection Ω of subsets of X, such that

(i) $\emptyset \in \Omega$ and $X \in \Omega$;

(ii) if $\{U_i\}_{i\in I} \subset \Omega$, then $\bigcup_{i\in I} U_i \in \Omega$;

(iii) if $U, V \in \Omega$, then $U \cap V \in \Omega$.

The sets in Ω are called **open sets** and their complements in X are called **closed sets**.

Note that point (iii) implies that any *finite* intersection of open sets is again open.

Definition 1.4. Let (X, Ω) be a topological space. A **neighbourhood** of a point $x \in X$ is a set N such that there exists $U \in \Omega$ with $x \in U \subset N$.

While formally a topological space consists of the pair (X, Ω), we usually omit mentioning Ω explicitly. Unless otherwise stated, when considering a metric space (X, d) we will always use the *metric topology*, i.e., the topology whose open sets are given by Definition 1.2.

Example 1.5. Different metric spaces can give rise to the same topology. In fact, any two norms on a finite-dimensional vector space give rise to the same topology. Consider, for example, $X = \mathbb{R}^n$ with the norms $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_i |x_i|$, and the corresponding distance functions $d_1(x, y) = \|x - y\|_1$ and $d_\infty(x, y) = \|x - y\|_\infty$. The norm inequalities

$$\|x\|_\infty \leq \|x\|_1 \leq n \cdot \|x\|_\infty$$

ensure that for any set $U \subset X$ and $x_0 \in U$, there is an open ball around x_0 in U with respect to one of these norms, if and only if there is one with respect to the other.

Specifying a topology is not always easy. Just like one can specify a vector space by giving a basis, one can also describe a topology in terms of a basis.

Definition 1.6. Let (X, Ω) be a topological space. A collection $\mathcal{B} \subset \Omega$ is called a **basis** for the topology Ω, if for all $U \in \Omega$ there exists a collection $\{B_i\}_{i\in I} \subset \mathcal{B}$ such that $\bigcup_{i\in I} B_i = U$. Given $x \in X$, a collection $\mathcal{B}$ is called a **neighbourhood basis** for x, if for every open set $U \in \Omega$ with $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Exercise 1.7. [1] Show that every metric space (X, d) is *first countable*: every point in X has a *countable* neighbourhood basis. Next, show that $\mathbb{R}$ with the *cofinite topology*, i.e., the topology whose open sets are the complements of finite sets, is not first countable. Hence, conclude that there are topological spaces that do not arise from a metric.

Example 1.8. The open intervals (a, b) form a basis of the metric topology on $\mathbb{R}$.

Given a collection of subsets $\mathcal{B}$ of a topological space (X, Ω), we say that $\mathcal{B}$ **generates** the topology if $\mathcal{B}$ is a basis of Ω. Any collection of subsets $\mathcal{B}$ that is closed under finite intersections generates a topology, whose open sets are just the unions of elements of $\mathcal{B}$.

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[1] The exercises in these notes are not examinable, they are only there for your enjoyment (though you may benefit intellectually from attempting them, consider them as "cross training").

Product spaces

Definition 1.9. Let X, Y be topological spaces. The **product topology** on $X \times Y$ is the topology generated by sets of the form $U \times V$, with $U \subset X$ open and $V \subset Y$ open.

Every open set in the product topology can be written as a (generally infinite) union of "rectangles" $U \times V$, but it is important to note that not all open sets are rectangles.

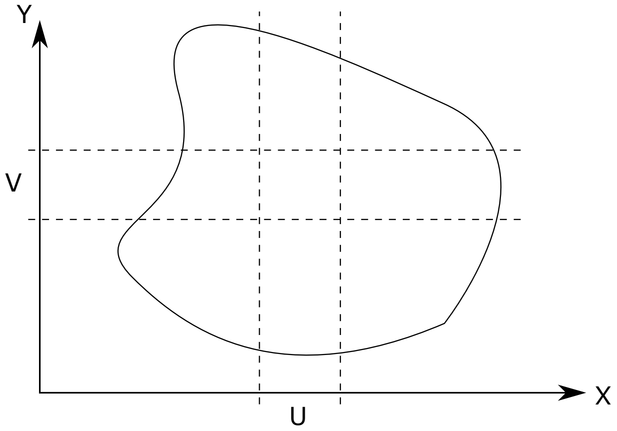


Figure 1.2: The product topology

Exercise 1.10. One can define the product topology on $\mathbb{R}^n$ recursively by setting $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ and $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ for $n \geq 2$. Show that the product topology on $\mathbb{R}^n$ is the same as the metric topology. (One can interpret the first as the topology generated by "open boxes", and the second as the topology generated by "open balls".)

Example 1.11. Just as in the case of $\mathbb{R}^n$, one can form the unit cube $I^n = I^{n-1} \times I$. The product topology in this case is also the same as the subspace topology.

Subspaces

Definition 1.12. Let (X, Ω) be a topological space and $A \subset X$ a subset. The **subspace topology** on A consists of the open sets

$$\Omega|_A = \{U \cap A \mid U \in \Omega\}.$$

Example 1.13. The closed interval $[0, 1] \subset \mathbb{R}$. Note that $(1/2, 1]$ is open in the subspace topology on $[0, 1]$!

Example 1.14. The **unit sphere**,

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}.$$

Note that the superscript denotes the dimension of the sphere, and not that of the ambient space in which the sphere lives.

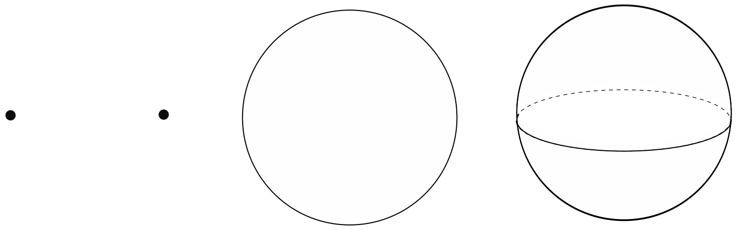


Figure 1.3: The spheres S^0, S^1 and S^2.

Example 1.15. The **unit ball**

$$\mathbb{D}^n = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 \leq 1\}.$$

Example 1.16. The **topological torus**, defined as product of 1-spheres (circles)

$$\mathbb{T}^1 = S^1, \quad \mathbb{T}^n = \mathbb{T}^{n-1} \times \mathbb{T}^1 = S^1 \times \cdots \times S^1 \text{ (n times)},$$

for $n \geq 2$. To justify the terminology “torus”, consider the parametrization of a torus X in $\mathbb{R}^3$ as the set of (x, y, z) such that

$$\begin{aligned} x(\theta, \varphi) &= (a\cos(\theta) + b)\sin(\varphi) \\ y(\theta, \varphi) &= (a\cos(\theta) + b)\cos(\varphi) \\ z(\theta, \varphi) &= a\sin(\theta). \end{aligned}$$

for $\theta, \varphi \in [0, 2\pi)$ and fixed $0 < a < b$.

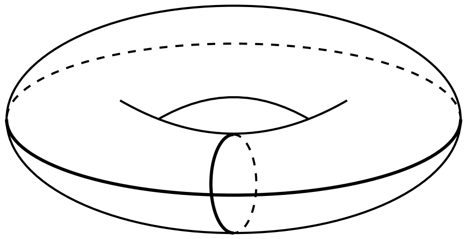


Figure 1.4: The embedded torus. The large circle going through the torus has radius b and the small circle bounding a section has radius a.

The product of spheres, $\mathbb{T}^2 = S^1 \times S^1$, can in turn be parametrized as the set of $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$ such that

$$x_1 = \cos(\theta),\ y_1 = \sin(\theta),\ x_2 = \cos(\varphi),\ y_2 = \sin(\varphi),$$

for $\theta, \varphi \in [0, 2\pi)$. This gives rise to a function

$$\begin{aligned} f \colon S^1 \times S^1 &\to X \\ \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} &\mapsto \begin{pmatrix} (ax_1 + b)y_2 \\ (ax_1 + b)x_2 \\ ay_1 \end{pmatrix} \end{aligned}$$

As we will see, this map is a continuous bijection with continuous inverse, also called a homeomorphism.

1.3 Topological equivalence

Definition 1.17. Let X, Y be topological spaces. A function $f\colon X \to Y$ is called **continuous**, if for any open set $V \subset Y$, the preimage $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in X.

We will refer to a continuous function simply as a **map**. Unless stated otherwise, all functions in this lecture are continuous.

Example 1.18. The identity function, $\mathrm{Id}_X\colon X \to X$, $x \mapsto x$, is clearly continuous, as is the inclusion $\iota\colon A \hookrightarrow X$ of a subset $A \subset X$ with the subspace topology.

Example 1.19. The function $\mathbb{R} \to S^1$ given by $t \mapsto (\cos(t), \sin(t))$ is continuous. We will often identify $\mathbb{R}^2$ with $\mathbb{C}$, and write e^{it} instead of $(\cos(t), \sin(t))$.

Example 1.20. The function $\mathbb{R} \to \mathbb{R}$, defined by

$$x \mapsto \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not continuous.

It is clear that compositions of continuous maps are continuous, a fact we will use repeatedly. Equally, the product map of two continuous maps is continuous. The following result, which will be used often, is a little less obvious.

Lemma 1.21. *(Pasting Lemma) Let $X = A \cup B$, with A, B both closed subspaces of a topological space X. Let $f\colon X \to Y$ be a function and assume that $f|_A$ and $f|_B$ are continuous. Then f is continuous.*

Exercise 1.22. Prove Lemma 1.21.

Definition 1.23. Let X, Y be topological spaces. A map $f\colon X \to Y$ is called a **homeomorphism**, if there exists a map $g\colon Y \to X$ such that

$$f \circ g = \mathrm{Id}_Y, \quad g \circ f = \mathrm{Id}_X.$$

If a homeomorphism between X and Y exists, these spaces are called **homeomorphic**, written $X \cong Y$.

Example 1.24. The identity Id_X is clearly a homeomorphism. The map $\mathbb{R} \to \mathbb{R}$, $x \to x^3$ is a homeomorphism, while $x \mapsto x^2$ is not (it is not invertible).

When we speak of spaces being “topologically equivalent”, we mean that they are homeomorphic. Topology does not distinguish between homeomorphic spaces.

Exercise 1.25. Show that the map $f\colon \mathbb{T}^2 \to X$ from Example 1.16 is a homeomorphism. This requires figuring out the inverse of this map and showing that both the map, and its inverse, are continuous.

1.4 The Fundamental Problem

The Fundamental Problem in topology is to classify spaces up to homeomorphism. More precisely, we would like to have a way of answering the question:

Given topological spaces X, Y, is $X \cong Y$?

Exercise 1.26. Show that the unit cube I^n is homeomorphic to the unit ball $\mathbb{D}^n$.

Example 1.27. $\mathbb{R}^0 = \{\mathrm{pt}\}$ (a single point) is not homeomorphic to $\mathbb{R}^1$ (a line).

How about $\mathbb{R}^1$ and $\mathbb{R}^2$? One might think that they are topologically not the same, as one is "somehow bigger". If they were homeomorphic, one could find a continuous and continuously invertible parametrization of the plane by a line. It turns out that the problem of showing that two real vector spaces of different dimension are not homeomorphic is not trivial. The tools developed in this module will allow to prove the following.

Theorem 1.28. *(Invariance of Domain, Brouwer 1910) $\mathbb{R}^m \cong \mathbb{R}^n$ if and only if $m = n$.*

Exercise 1.29. Try to show that $S^2 \ncong \mathbb{T}^2$.

To show that two spaces are homeomorphic, one only needs to provide a homeomorphism. To show that they are *not* homeomorphic is more difficult, and amounts to finding a property that is a) invariant under homeomorphism, and b) is satisfied by one of the spaces but not the other. As we will see, algebraic invariants such as the fundamental group (the main topic of this module) allow to accomplish this.

Lecture 2

In this lecture we will have a closer look at the construction of topological spaces using disjoint unions and quotient spaces, and show how to formalize “cut and paste” operations on topological spaces.

2.1 The disjoint union

Definition 2.1. Let X, Y be topological spaces. The **disjoint union** of X and Y is the topological space with underlying set

$$X \sqcup Y = X \times \{0\} \cup Y \times \{1\},$$

and the topology whose basis consists of sets of the form $U \times \{0\}$ and $V \times \{1\}$ for $U \subset X$ and $V \subset Y$ open.

Remark 2.2. The $\{0\}$ and $\{1\}$ in the definition are arbitrary; there is not deeper meaning in using these two numbers. Formally, given a topological space X and a one-point space $\{\mathrm{pt}\}$ (we often use this notation to denote a space consisting of only one point, whose precise identity does not matter for topological purposes), we can construct the space $X \times \{\mathrm{pt}\}$ and equip it with the product topology. It is then an easy exercise to show that the inclusion map

$$\iota \colon X \hookrightarrow X \times \{\mathrm{pt}\}, \quad x \mapsto (x, \mathrm{pt})$$

is a homeomorphism. We will therefore not distinguish these two spaces, and think of $X \times \{0\}$ and $X \times \{1\}$ simply as two disjoint copies of X. For example, $X \sqcup X$ amounts to taking the union of X and a disjoint copy of X, while $X \cup X$ is just X. We can iterate this construction and write, for a collection of topological spaces $\{X_i\}_{i \in \mathbb{Z}}$,

$$\bigsqcup_i X_i = \bigcup_i X_i \times \{i\}.$$

For example, if we take $X_i = [i, i+1)$, the half open interval starting at i, then $\bigcup_i X_i = \mathbb{R}$.

Example 2.3. The 0-sphere can be written as $S^0 \cong \{\mathrm{pt}\} \sqcup \{\mathrm{pt}\}$. The topology on this space is the discrete topology (all subsets are open).

2.2 The quotient topology

Another important construction is the quotient, which formalizes the notion of “glueing” or “pasting”. Recall that an equivalence relation is a subset $E \subset X \times X$ such that:

- for all $x \in X$, $(x, x) \in E$ (reflexive);
- if $(x, y) \in E$, then $(y, x) \in E$ (symmetric);
- if $(x, y) \in E$ and $(y, z) \in E$, then $(x, z) \in E$ (transitive).

Once we fix an equivalence relation E, we usually write $x \sim y$ instead of $(x, y) \in E$. The **equivalence class** of $x \in X$ is the set

$$[x] = \{y \in X \mid x \sim y\}.$$

The set of all equivalence classes of an equivalence relation E is denoted by ${}^{X}/_{E}$ or ${}^{X}/_{\sim}$. The **quotient map** is the map $q \colon X \to {}^{X}/_{E}$, $q(x) = [x]$.

Definition 2.4. The **quotient topology** on ${}^{X}/_{E}$ has as open sets those $V \subset {}^{X}/_{E}$ for which $q^{-1}(V) = \{x \in X \mid q(x) \in V\}$ is open.

Note that a subset $V \subset {}^{X}/_{E}$ is open if and only if

$$U = \bigcup_{[x] \in V} [x]$$

is open in X. By definition, the quotient map is continuous.

Exercise 2.5. Let X be a topological space, E an equivalence relation, and ${}^{X}/_{E}$ the corresponding quotient space. Show that for all topological spaces Z and all functions $g \colon {}^{X}/_{E} \to Z$, g is continuous if and only if the composition $f = g \circ q$ is continuous.

$$\begin{tikzcd}
X \arrow[d, "q"] \arrow[dr, "f"] & \\
{}^{X}/_{E} \arrow[r, "g"] & Z
\end{tikzcd}$$

Example 2.6. Consider $X = I = [0, 1]$. Define the equivalence relation

$$x \sim y \Leftrightarrow (x = y) \text{ or } (x = 0, y = 1) \text{ or } (x = 1, y = 0).$$

The quotient space ${}^{X}/_{E}$ then consists of the classes $[x] = \{x\}$ for $x \notin \{0, 1\}$ and $[0] = [1] = \{0, 1\}$. The result is an interval with the endpoints “glued together”, sometimes written ${}^{I}/_{(0\,\sim\,1)}$ to highlight the fact that only 0 and 1 are identified. If we parametrize the circle by

$$f \colon I \to S^1, \quad t \mapsto \exp(2\pi i t),$$

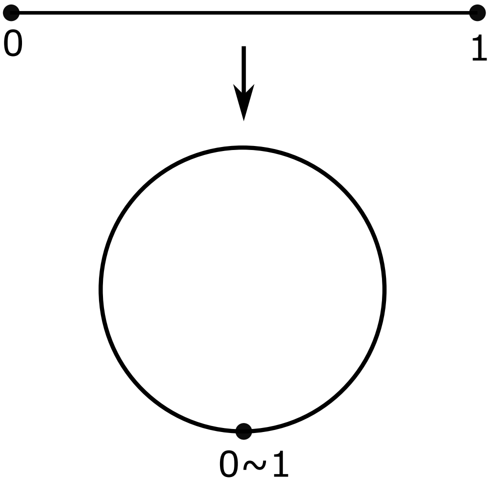


Figure 2.5: Glueing an interval at the endpoints to obtain a circle.

then this map is one-to-one except at the endpoints, and identifying these endpoints gives rise to a homeomorphism (recall our convention of viewing the circle as subset $S^1 \subset \mathbb{C}$). One also says that the map f "factors" over the quotient space, as indicated in the following diagram:

$$\begin{tikzcd}
I \arrow[d, "q"] \arrow[dr, "f"] & \\
{}^{I}/_{(0 \sim 1)} \arrow[r, "\cong"] & S^1
\end{tikzcd}$$

The above example is a special case of a more general construction. Let $A \subset X$ be a subset. Such a subset gives rise to the equivalence relation

$$x \sim y \Leftrightarrow (x = y) \text{ or } \{x, y\} \subset A.$$

The corresponding quotient space, by some abuse of notation sometimes referred to as ${}^{X}/_{A}$ (note that A is a subset of X, and not a subset of $X \times X$!), consists of classes $[x] = \{x\}$ if $x \in X - A$ and $[x] = [y]$ if $x, y \in A$. In words, ${}^{X}/_{A}$ corresponds to "crushing" the set A onto one point.

Exercise 2.7. Recall the set inclusion $S^{n-1} \subset \mathbb{D}^n$. Show that ${}^{\mathbb{D}^n}/_{S^{n-1}} \cong S^n$. Since $[0,1] \cong [-1,1] = \mathbb{D}^1$ and $\{0,1\} \cong \{-1,1\} = S^0$, this generalizes Example 2.6. Can you interpret the case $n = 2$ visually?

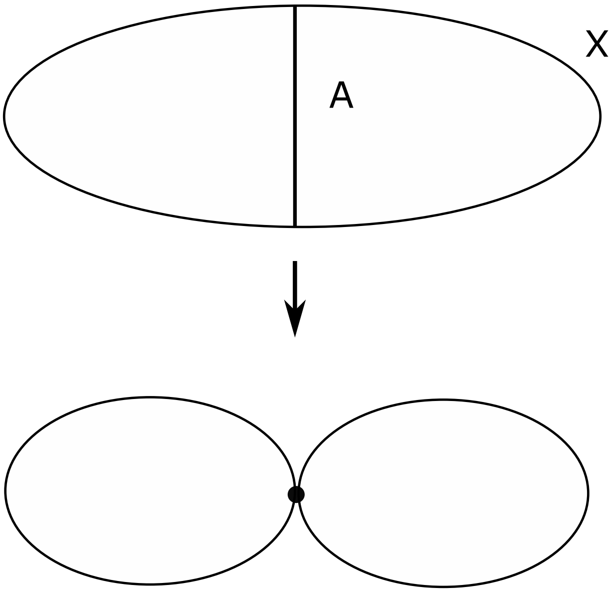
Disjoint unions and quotients can be combined to construct new spaces.

Example 2.8. Consider the disjoint union of $\mathbb{Z}$ times the interval I. Formally, this amounts to taking the union

$$X = \bigcup_{k \in \mathbb{Z}} I \times \{k\}.$$

By gluing the endpoints, i.e., identifying $\{1\} \times \{k\}$ with $\{0\} \times \{k+1\}$, one obtains a set that is homeomorphic to $\mathbb{R}$ (check this!)

In a similar fashion, one can build up $\mathbb{R}^2$ by tiling more sophisticated shapes.



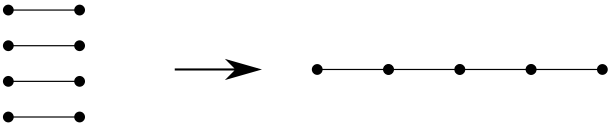


Figure 2.6: The real line by gluing intervals.

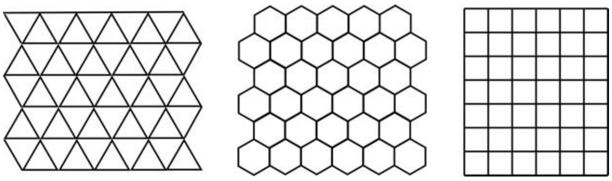
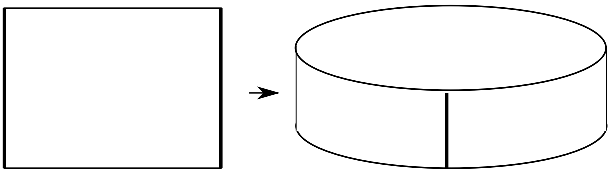


Figure 2.7: The plane by tiling shapes.

Example 2.9. Consider the square $X = I^2 = [0,1]^2$. Define an equivalence relation

$$(x,y) \sim (x',y') \Leftrightarrow (x,y) = (x',y') \text{ or } (y = y', \{x,x'\} \subset \{0,1\}).$$

In words, we identify each point on the left boundary with the corresponding point on the right boundary. The result is a space homeomorphic to a cylinder.



Example 2.10. Consider again the square and define an equivalence relation by

$$(x,y) \sim (x',y') \Leftrightarrow (x,y) = (x',y') \text{ or } (y = 1-y', \{x,x'\} \subset \{0,1\}).$$

We now identify each point on the left boundary with coordinate y with the point on the right boundary with coordiante $1 - y$.

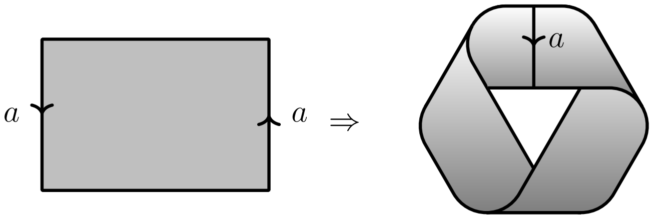


Figure 2.8: The Möbius strip

The result of this construction is the famous Möbius strip, a surface with only one side. This surface is *not* homeomorphic to the cylinder (try to find out why!)

Exercise 2.11. Show that

$$\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}^2,$$

where $\mathbb{R}^2/\mathbb{Z}^2$ is the quotient space with respect to the equivalence relation

$$(x, y) \sim (x', y') \Leftrightarrow x - x' \in \mathbb{Z},\ y - y' \in \mathbb{Z}.$$

The quotient map is an example of an **identification map**: this is a surjective map between topological spaces, $f\colon X \to Y$, such that $U \subset Y$ is open if and only if $f^{-1}(U)$ is open. Identification maps satisfy the following property.

Proposition 2.12. *A surjective map $f\colon X \to Y$ is an identification map if and only if for every space Z and every function $g\colon Y \to Z$, $g \circ f$ is continuous if and only if g is continuous.*

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Exercise 2.13. Prove Proposition 2.12.

Lecture 3

So far we looked at the notion of homeomorphism, and considered spaces to be "topologically equivalent" if they are homeomorphic. Intuitively, homeomorphic spaces are the same up to "stretching" and "shrinking", but not crushing or cutting. Homeomorphism is a rather fine equivalence, and considering a coarser relation such as homotopy equivalence can be useful. We begin by discussing retractions, continuous functions of a space to a subspace, and deformation retracts, which formalize the idea of *continuously* squeezing a space onto a subspace. This concept will lead naturally to the idea of homotopy.

3.1 Retractions

Definition 3.1. A **pair** of spaces (X, A) consists of a topological space X and a subspace $A \subset X$. If $A = \{x\}$, then we write (X, x) and call this a **pointed** space.

Example 3.2. Consider the pair of spaces $(\mathbb{R}^2 - \{0\}, S^1)$, or the pair $(S^1, (1, 0))$.

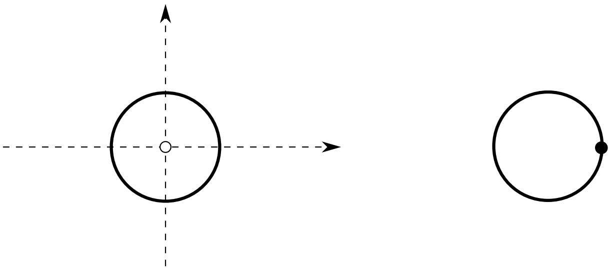


Figure 3.9: The pairs $(\mathbb{R}^2 - \{0\}, S^1)$ and $(S^1, (1, 0))$.

Definition 3.3. A subset $A \subset X$ is a **retract** of X if there is a map $r\colon X \to A$ (the **retraction**) such that the restriction satisfies

$$r|_A = \mathrm{Id}_A,$$

i.e., $r(a) = a$ for $a \in A$.

Example 3.4. The set $\mathbb{R}^2 - \{0\}$ retracts to S^1 via $r(x) = x/\|x\|$.

Exercise 3.5. Show that $X = I$ *does not* retract to $A = \{0, 1\}$.

The following generalization is non-trivial, and we will only be able to prove it later after discussing the fundamental group.

Theorem 3.6. *(Brouwer) The ball $\mathbb{B}^n$ does not retract to S^{n-1}.*

We next describe what it means to *deform* a space onto a subspaces in a continuous manner.

Definition 3.7. Let (X, A) be a pair of spaces. X **deformation retracts** to A (and A is called a **deformation retract** of X), if there exists a one-parameter family of functions $f_t \colon X \to X$, $t \in I = [0, 1]$, such that

$$f_0 = \mathrm{Id}_X, \quad f_1(X) = A, \quad f_t|_A = \mathrm{Id}_A, t \in [0, 1],$$

and the map $X \times I \to X$, $(x, t) \mapsto f_t(x)$ is continuous.

In the literature, this notion is sometimes called a *strong* deformation retract, the strong referring to the requirement that $f_t|_A = \mathrm{Id}_A$ throughout.

Example 3.8. $\mathbb{R}^n$ deformation retracts to 0 by means of $f_t(x) = (1 - t)x$. This is called the **straight-line** homotopy.

Exercise 3.9. Show that $\mathbb{R}^n - \{0\}$ deformation retracts to S^{n-1} via $f_t(x) = (1 - t)x + tx/\|x\|$.

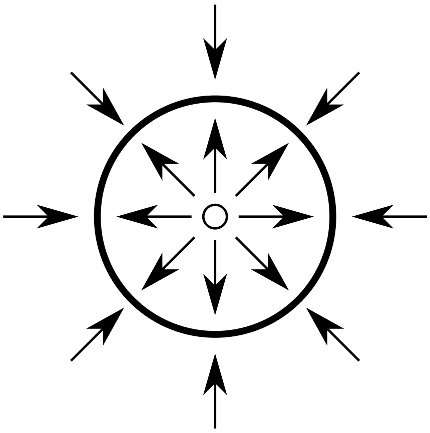


Figure 3.10: Deformation of a punctured plane $\mathbb{R}^2 - \{0\}$ onto the circle S^1.

3.2 Homotopy

Definition 3.10. Let X, Y be topological spaces, and $I = [0, 1]$. A map

$$F \colon X \times I \to Y$$

is called a **homotopy**. If $f_t(x) = F(x, t)$, then F is called a homotopy from f_0 to f_1. We say that two maps f, g are **homotopic**, written $f \simeq g$, if there exists a homotopy F such that $f_0 = f$ and $f_1 = g$.

Proposition 3.11. *Homotopy is an equivalence relation: if $f, g, h \colon X \to Y$ are maps, then*

(i) $f \simeq f$;

(ii) $f \simeq g \Leftrightarrow g \simeq f$;

(iii) $f \simeq g, g \simeq h \Rightarrow f \simeq h$.

Proof of Proposition 3.11. The points (i) and (ii) are utterly trivial. For (iii), assume we have homotopies $F\colon X \times I \to Y$ and $G\colon X \times I \to Y$ such that $f_0 = f$, $f_1 = g = g_0$, and $g_1 = h$. Construct a new homotopy between f and h as

$$H(x,t) = \begin{cases} F(x, 2t) & t \leq 1/2, \\ G(x, 2t-1) & t \geq 1/2. \end{cases}$$

The continuity of H follows from the Pasting Lemma (Lemma 1.21 in Lecture 2). □

Remark 3.12. There is nothing special about using the interval I in the definition of homotopy: one could equally use any other closed interval $[a,b]$. In fact, let $\varphi\colon I \to [a,b]$ be the continuous reparametrization given by $\varphi(t) = a+(b-a)t$. Then given a continuous map $F\colon X \times [a,b] \to Y$ we can get a homotopy $G\colon X \times I \to Y$ by setting $G(x,t) = F(x, \varphi(t))$.

Definition 3.13. Let X, Y be topological spaces. We say X is **homotopy equivalent** to Y (written $X \simeq Y$), if there are maps

$$f\colon X \to Y, \quad g\colon Y \to X,$$

such that

$$g \circ f \simeq \mathrm{Id}_X, \quad f \circ g \simeq \mathrm{Id}_Y.$$

Homotopy equivalence allows for squeezing spaces, but not for tearing. Note that homeomorphic spaces are homotopy equivalent, but the converse does not hold. In particular, if we manage two show that two spaces are not homotopy equivalent, then they cannot be homeomorphic.

Example 3.14. The space $\mathbb{R}^n$ is homotopy equivalent to a point for any n, $\mathbb{R}^n \simeq \mathbb{R}^0$. To see this, consider the maps

$$f\colon \mathbb{R}^n \to \mathbb{R}^0, \quad x \mapsto 0$$

and

$$g\colon \mathbb{R}^0 \to \mathbb{R}^n, \quad 0 \mapsto 0.$$

Note that $f \circ g = \mathrm{Id}_{\mathbb{R}^0}$. Consider now the map $g \circ f\colon \mathbb{R}^n \to \mathbb{R}^n$, mapping any x to 0. Consider the straight-line homotopy

$$F\colon \mathbb{R}^n \times I \to \mathbb{R}^n, \quad (x,t) \mapsto tx.$$

Then $f_0 = g \circ f$ and $f_1 = \mathrm{Id}_{\mathbb{R}^n}$, which shows that $g \circ f \simeq \mathrm{Id}_{\mathbb{R}^n}$.

Exercise 3.15. Show that $\simeq$ is an equivalence relation on topological spaces.

Example 3.16. For all m, n, we have $\mathbb{R}^n \simeq \mathbb{R}^m$. More generally, for all topological spaces X, we have $X \times \mathbb{R}^n \simeq X$.

Example 3.17. We have $\mathbb{R}^m - \{0\} \simeq S^{m-1}$. To construct the homotopy, consider the inclusion map $f\colon S^{m-1} \to \mathbb{R}^m - \{0\}$, and the retraction $g\colon \mathbb{R}^m - \{0\} \to S^{m-1}$, given by $x \mapsto x/\|x\|$. Then $g \circ f = \mathrm{Id}_{S^{m-1}}$, and for $f \circ g$ we construct the homotopy

$$F\colon \mathbb{R}^m - \{0\} \times I \to \mathbb{R}^m - \{0\}, \quad (x,t) \mapsto (1-t)x + tx/\|x\|.$$

Note that this is just the deformation retract from Example 3.9, and that $f_0 = \mathrm{Id}_{\mathbb{R}^n - \{0\}}$ and $f_1 = g \circ f$.

How does homotopy relate to the notion of a deformation retract? Clearly, if a space X deformation retracts to a point $x \in X$, then it is homotopy equivalent to a point (contractible), but the converse need not hold! It is, for example, non-trivial to show that spheres S^n for $n \geq 1$ are not contractible.

Definition 3.18. A topological space X is called **contractible** if $X \simeq \{\mathrm{pt}\}$.

We have seen that $\mathbb{R}^n$ for $n \geq 1$ is contractible: there exist maps $f\colon \mathbb{R}^n \to \mathbb{R}^0$ and $g\colon \mathbb{R}^0 \to \mathbb{R}^n$ such that $f \circ g = \mathrm{Id}_{\mathbb{R}^0}$ and $g \circ f \simeq \mathrm{Id}_{\mathbb{R}^n}$, with the homotopy from the identity $\mathrm{Id}_{\mathbb{R}^n}$ to the retraction $g \circ f\colon \mathbb{R}^n \to \{0\}$ given by the *linear homotopy* $F(x,t) = (1-t)x$.

Remark 3.19. One should compare the notion of contractibility with that of a deformation retract. If X deformation retracts to a point $A = \{a\} \subset X$, then X is contractible. To be more precise, by the definition of a deformation retract, we have a family of maps $f_t\colon X \to X$ such that $f_0 = \mathrm{Id}_X$, $f_1(x) = a$ for $x \in X$, and $f_t(a) = a$ for all t. Considering the retraction $f\colon X \to A$ and the inclusion $g\colon A \to X$, we get $f \circ g = \mathrm{Id}_A$. On the other hand, $g \circ f = f_1$ and $\mathrm{Id}_X = f_0$, so that we get a homotopy $\mathrm{Id}_X \simeq g \circ f$ by setting $F(x,t) = f_t(x)$.

The converse is **not true**: a spaces can be contractible but not deformation retract to a point (try to think of an example!)

Theorem 3.20. *The sphere S^n for $n \geq 0$ is not contractible.*

Exercise 3.21. Proof the above Theorem for $n = 0$.

Theorem 3.20 is highly non-trivial. The case $n = 1$ will occupy a large part of this course. The general case follows from homology theory, which is the subject of follow-up courses in Algebraic Topology.

Lecture 4

We have seen different notions of equivalence: homeomorphism ($X \cong Y$), homotopy of maps ($f \simeq g$) and homotopy equivalence of spaces ($X \simeq Y$). The last of these notions allows spaces to be identified that look superficially different, but can somehow be deformed or "continuously collapsed" into one another. In this lecture we will study paths and loops as a way to understand the topology of spaces. Recall that a **map** is always assumed to be continuous.

4.1 Paths

Definition 4.1. Let $x, y \in X$. A **path** from x to y is a map $f\colon I \to X$ with $f(0) = x$ and $f(1) = y$.

Definition 4.2. Let $f, g\colon I \to X$ be paths with $f(1) = g(0)$. The path $f * g\colon I \to X$, defined by

$$f * g(t) = \begin{cases} f(2t) & t \leq 1/2 \\ g(2t-1) & t \geq 1/2. \end{cases}$$

is called the **concatenation** of f and g.

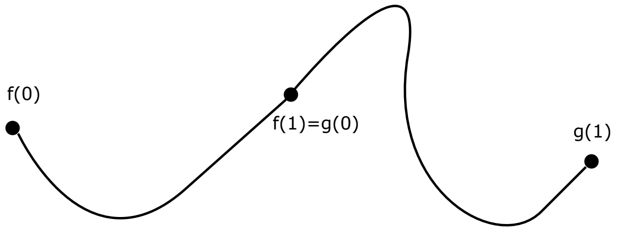


Figure 4.11: The concatenation of two paths.

Remark 4.3. It is **very important** to note that a path is not the same as its image! Consider, for example, the path $f\colon I \to S^1$, $t \mapsto \exp(4\pi i t)$. The image of this path is the circle S^1 (considered as subset of $\mathbb{C}$), but as a path it goes around the circle twice. In particular, a path need not be injective.

Definition 4.4. A topological space X is called **path connected** if for any two points $x, y \in X$ there exists a path $f\colon I \to X$ with $f(0) = x$ and $f(1) = y$.

Since paths are maps between topological spaces, we can consider *homotopies* of paths: given two paths $f, g\colon I \to X$, a homotopy is given by a map $F\colon I \times I \to X$ with $f_0 = f$ and $f_1 = g$.

Exercise 4.5. Show that if X is path connected, then every path $f\colon I \to X$ is homotopic to a constant path $g(t) = x$.

To get more useful topological information out of paths, we consider paths with common endpoints.

Definition 4.6. Let $x, y \in X$ and let $f, g\colon I \to X$ be paths from x to y. Then f is homotopic to g relative to the boundary (or relative to the endpoints), written $f \overset{\partial}{\simeq} g$, if there is a homotopy

$$F\colon I \times I \to X$$

with $f_0 = f$, $f_1 = g$ and for all t, $f_t(0) = x$, $f_t(1) = y$.

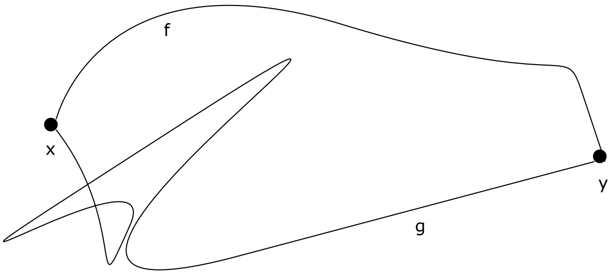


Figure 4.12: Homotopic relative to endpoints.

Example 4.7. Let $X = S^1 \subset \mathbb{C}$ and consider the maps $f(t) = \exp(\pi i t)$ and $g(t) = \exp(-\pi i t)$. Thus $f(t)$ moves from 1 to -1 along the top, while $g(t)$ moves from 1 to -1 along the bottom half of the circle.

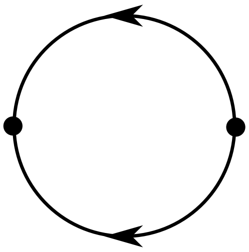


Figure 4.13: Two paths that are homotopic, but where there is no homotopy in S^1 that preserves endpoints.

Then $f \simeq g$ (construct an explicit homotopy!), but not in an end-point preserving fashion. While constructing a homotopy between these paths is easy, showing that this can not be done in an end-point preserving way is surprisingly hard!

Lemma 4.8. *Let $x, y \in X$. The relative homotopy $\overset{\partial}{\simeq}$ is an equivalence relation on the set of paths $I \to X$ with endpoints x, y.*

Proof. It is clear that $f \overset{\partial}{\simeq} f$ and $f \overset{\partial}{\simeq} g \Leftrightarrow g \overset{\partial}{\simeq} f$ for paths $f, g\colon I \to X$ with common endpoints. To show transitivity, let $f, g, h\colon I \to X$ be paths from x to y such that $f \overset{\partial}{\simeq} g$ and $g \overset{\partial}{\simeq} h$. This means that there are homotopies

$$F\colon I \times I \to X, \quad G\colon I \times I \to X$$

such that $f_0 = f$, $f_1 = g_0 = g$, and $g_1 = h$. Define a new map $H\colon I \times I \to X$ by

$$H(s,t) = \begin{cases} F(s,2t) & \text{if } t \le 1/2 \\ G(s,2t-1) & \text{if } t \ge 1/2 \end{cases}.$$

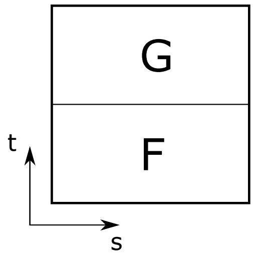


Figure 4.14: The homotopy H coincides with (a reparametrized version of) F on the lower rectangle ($t \le 1/2$), and with G on the upper rectangle ($t \ge 1/2$).

Clearly, $h_0 = f$ and $h_1 = h$. Moreover, by the Pasting Lemma 1.21, H is continuous, which shows that $f \overset{\partial}{\simeq} h$. ☐

Lemma 4.9. *Assume that* $f \overset{\partial}{\simeq} g$ *and* $f' \overset{\partial}{\simeq} g'$*, where* $f, g\colon I \to X$ *are paths with* $f(1) = g(0)$*. Then* $f * f' \overset{\partial}{\simeq} g * g'$*.*

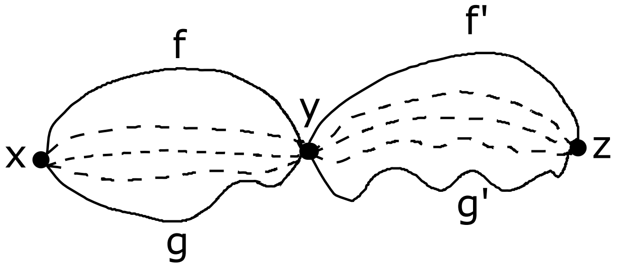


Figure 4.15: Concatenated homotopies.

Proof. The proof is essentially the same as that of Lemma 21.1, but with the role of s and t reversed. The situation is visualized as follows.

Formally, consider homotopies F and F' with $f_0 = f$, $f_1 = g$, $f'_0 = f'$, $f'_1 = g'$. Define a new map

$$G(s,t) = \begin{cases} F(2s,t) & \text{if } s \le 1/2 \\ G(2s-1,t) & \text{if } s \ge 1/2 \end{cases}.$$

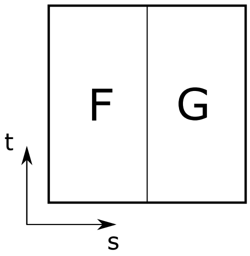


Figure 4.16: The homotopy H coincides with (a reparametrized version of) F on the left rectangle ($s \leq 1/2$), and with G on the right rectangle ($s \geq 1/2$).

As before, this map is continuous, and satisfies $h_0 = f * f'$ and $h_1 = g * g'$, thus showing that $f * f' \overset{\partial}{\simeq} g * g'$. $\square$

In the coming lecture we will look at special types of paths, called **loops**, which start and end at the same point. Using Lemma 21.1 and 21.2, we will see that the set of equivalence classes of loops have a group structure, leading to the concept of the **Fundamental Group** of a pointed topological space.

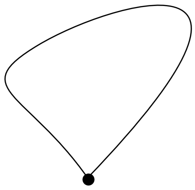
Lecture 5

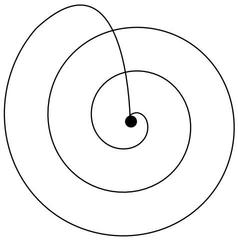
In this lecture we look at loops and will discover that there is an underlying algebraic structure, the Fundamental Group, that allows to gain insight into the topological features of spaces.

5.1 Loops and the Fundamental Group

We call a topological space X with a point x_0 a **pointed space** (X, x_0).

Definition 5.1. Let (X, x_0) be a pointed space. A **loop** is a path $f\colon I \to X$ with $f(0) = f(1) = x_0$.





Note that the composition of two loops is again a loop.

Since in Lecture 4 we have seen that homotopy on paths with common endpoints is an equivalence relation, we can form equivalence classes of loops

$$[f] = \{g\colon g\colon I \to X, g(0) = g(1) = x_0, g \overset{\partial}{\simeq} f\}.$$

The set of equivalence classes of loops on (X, x_0) is denoted by $\pi_1(X, x_0)$. We have seen that if $f \simeq g$ and $f' \simeq g'$, then $f * f' \simeq g * g'$. From this it follows that the equivalence class $[f * f']$ only depends on the classes $[f]$ and $[f']$, and not on the particular choice of representative in each class. This allows us to define a product

$$[f] \bullet [g] := [f * g].$$

Proposition 5.2. $(\pi_1(X, x_0), \bullet)$ *is a group, called the* ***Fundamental Group*** *of the pointed space* (X, x_0)*. The unit element is the class* $[e]$ *of the constant loop, and for every* $[f]$*, the inverse* $[f]^{-1}$ *is the class* $[\overline{f}]$*, where* $\overline{f}(t) = f(1-t)$ *is the inverse loop.*

Proof. **(1)** We first show that $[f] \bullet [e] = [e] \bullet [f] = [f]$. If $f\colon I \to X$ is a loop, we thus need to show that

$$f * e \overset{\partial}{\simeq} e * f \overset{\partial}{\simeq} f.$$

For this, we first construct a homotopy from $f * e$ to $e * f$ as follows

$$F(s,t) = \begin{cases} x_0 & 2s \leq t \text{ or } 2s - 1 \geq t \\ f(2s-t) & \text{else.} \end{cases}$$

Indeed, we see that

$$f_0(s) = F(s,0) = \begin{cases} f(2s) & \text{if } s \leq 1/2 \\ x_0 & \text{if } s \geq 1/2, \end{cases}$$

which is the definition of $f * e$. Similarly, one checks that f_1 coincides with $e * f$. By the Pasting Lemma 1.21, we get a homotopy. How does one derive this homotopy? A simple way is to draw a diagram to visualise what is happening.

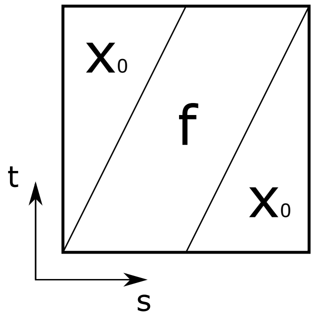


Figure 5.17: The composition of f and the constant loop on the lower boundary, and the composition of the constant loop and f on the upper boundary. The parameter t on the vertical axis parametrizes the different maps f_t in the homotopy.

In Figure 5.17, one then only needs to figure out the regions in (s,t) space where the homotopy coincides with f, and where it coincides with x_0. Of course, the functions should then be rescaled so that every vertical slice of of the f-band starts with $f(0)$ and ends with $f(1)$. To show that $f \overset{\partial}{\simeq} e * f$, one similarly uses a diagram

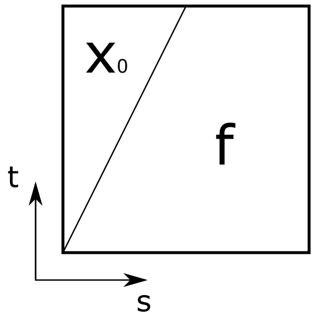


Figure 5.18: The loop f is homotopic to the composition $e * f$.

We leave the explicit description (and that for $f \overset{\partial}{\simeq} f * e$) as an exercise.

(2) We next show the existence of the inverse. Let $f\colon I \to X$ be a loop and $\overline{f}\colon I \to X$ the loop with $\overline{f}(s) = f(1-s)$. We need to show that $\overline{f} * f \overset{\partial}{\simeq} e$ and $f * \overline{f} \overset{\partial}{\simeq} e$. For this, we consider the diagram

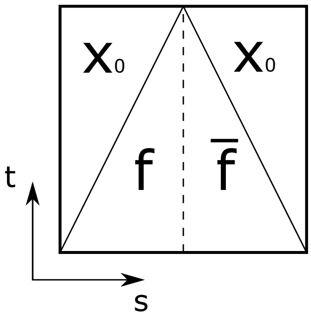


Figure 5.19: A loop composed with its inverse is homotopic to the constant loop.

Explicitly, this diagram suggests the homotopy

$$F(s,t) = \begin{cases} x_0 & \text{if } 2s \leq t \text{ of } 2-2s \leq t \\ f(2s-t) & \text{if } 2s \leq 1 \text{ and } t \leq 2s \\ f(2s+t-1) & \text{if } 2s \geq 1 \text{ and } t \leq 2s-2. \end{cases}$$

One verifies that $f_0 = f * \overline{f}$ and that $f_1 = e$. The same diagram with $\overline{f}$ and f interchanged gives a homotopy $\overline{f} * f \overset{\partial}{\simeq} e$.

(3) To verify associativity, namely that $(f*g)*h \overset{\partial}{\simeq} f*(g*h)$, we use the diagram

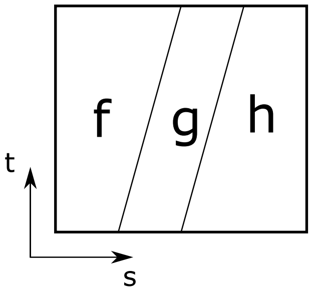


Figure 5.20: Three loops composed in different order.

The detailed expression for the homotopy is left as an exercise. □

Example 5.3. $\pi_1(\mathbb{R}^2, \{0\}) = 0$, as every loop is homotopic to the origin.

Example 5.4. Let $X = S^1 \subset \mathbb{C}$ with $x_0 = 1$. Then intuitively one would say $\pi_1(X, x_0) \cong \mathbb{Z}$, when looking at loops of the form $f(t) = \exp(2\pi kit)$ for $k \in \mathbb{Z}$, that is, it goes around k times in clockwise or anti-clockwise direction. The composition of two such loops is clearly another such loop. On the unit circle, this composition can be realized as the product $\exp(2\pi kit) \cdot \exp(2\pi mit) = \exp(2\pi i(k+m)t)$. While this example is intuitively clear, it is not a formal proof! The problem is that the fundamental group consists of *homotopy classes* of loops, not loops themselves. If we only consider loops of the form above, then clearly we have a group, but how can we be sure that *every* loop is homotopic to a loop of this form? Also, how can we be sure that two such loops with different parameter k are not homotopic to each other or to the constant loop? We will get back to this example after discussing covering spaces, and formally prove that $\pi_1(S^1, x_0) \cong \mathbb{Z}$.

Lecture 6

In this lecture we discuss an important class of spaces, where any two points can be connected by a path.

6.1 Path connected spaces

For certain spaces, $\pi_1(X, x_0)$ is actually a topological invariant of the space itself: it does not depend on the choice of basepoint.

Proposition 6.1. *If X is path-connected, then for any two $x_0, x_1 \in X$, the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof. Let $h\colon I \to X$ be a path from x_0 to x_1, with inverse path $\overline{h}$. Define the map

$$\begin{aligned} \beta_h\colon \pi_1(X, x_0) &\to \pi_1(X, x_1) \\ [f] &\mapsto [\overline{h} * f * h]. \end{aligned}$$

We need to show that β_h is an isomorphism of groups, with $\beta_{\overline{h}} = \beta_h^{-1}$ as inverse.

(1) We first show that β_h is a bijection. Note that since $h * \overline{h} \overset{\partial}{\simeq} e_{x_0}$ (the constant loop on x_0) and $\overline{h} * h \overset{\partial}{\simeq} e_{x_1}$, we get that

$$\beta_{\overline{h}} \circ \beta_h([f]) = [h * \overline{h} * f * h * \overline{h}] = [f],$$

and hence $\beta_{\overline{h}} \circ \beta_h = \mathrm{Id}_{\pi_1(X,x_0)}$. Similarly, one shows that $\beta_h \circ \beta_{\overline{h}} = \mathrm{Id}_{\pi_1(X,x_1)}$, thus showing that β_h is a bijection with inverse map $\beta_{\overline{h}}$.

(2) We next need to verify that β_h is a group homomorphism. This is the case, because

$$\begin{aligned} \beta_h([f] \bullet [g]) &= \beta_h([f * g]) \\ &= [\overline{h} * f * g * h] \\ &= [\overline{h} * f * h * \overline{h} * g * h] \\ &= [\overline{h} * f * h] \bullet [\overline{h} * g * h] \\ &= \beta_h([f]) \bullet \beta_h([g]). \end{aligned}$$

The same argument works for $\beta_{\overline{h}}$, showing that we have an isomorphism. □

For path-connected spaces X, we will often simply write $\pi_1(X)$ if we only care about the structure of the group and not the basepoint.

6.2 The Fundamental Group and the Fundamental Theorem

In Lecture 5, we heuristically argued that $\pi_1(S^1) \cong \mathbb{Z}$, but did not provide a formal prove. While intuitively clear, this result is not completely obvious to proof. As usual, we consider the circle $S^1 \subset \mathbb{C}$ as subset of the complex numbers, and define the loops with basepoint 1

$$\omega_n \colon I \to S^1, \quad \omega_n(s) = \exp(2\pi i n \cdot s)$$

for $n \in \mathbb{Z}$. Thus each ω_n goes around the circle $|n|$ times in counterclockwise (if $n > 0$) or clockwise (if $n < 0$) direction. In particular, $\omega_0 = e$ is the constant loop with basepoint 1. Note that $\omega_n = \omega_1^n$. The following theorem states the (apparently, but not, obvious) fact that every loop on S^1 with basepoint 1 is of this form.

Theorem 6.2. *The fundamental group* $\pi_1(S^1, 1)$ *is the infinite cyclic group generated by* $[\omega_1]$, *i.e.,*

$$\pi_1(S^1, 1) = (\{[\omega_n] \mid n \in \mathbb{Z}\}, \bullet) \cong \mathbb{Z}.$$

To prove this theorem, we have to make an excursion and discuss covering spaces. Before embarking on this theory, we will show that this result allows to prove the Fundamental Theorem of Algebra [2]

Theorem 6.3. *Every non-constant, complex polynomial* $p(z) \in \mathbb{C}[z]$ *has at least one complex root, i.e., a* $\lambda \in \mathbb{C}$ *such that* $p(\lambda) = 0$.

Proof. The proof is by contradiction. Assume that there exists a polynomial

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

of degree $n \geq 1$ such that $p(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$ (we can without lack of generality assume $p(z)$ to be *monic*, meaning that the coefficient of z^n is 1). For every real number $r > 0$, such a polynomial gives rise to a loop g_r on S^1

$$g_r(s) = \frac{p(r \exp(2\pi i s))/p(r)}{|p(r \exp(2\pi i s))/p(r)|},$$

with basepoint $g_r(0) = g_r(1) = 1$. The strategy of the proof is two show, via two different homotopies, that

$$g_r \overset{\partial}{\simeq} e, \qquad \text{and } g_r \overset{\partial}{\simeq} \omega_n.$$

Of course this should not be possible, giving the desired contradiction.

(1) Consider the homotopy $f_t = g_{tr}$. Then $f_1 = g_r$ and $f_0 = e$, and $[g_r] = [e]$.

(2) To show that g_r is homotopic to ω_n, the idea is to use p to construct a sequence of polynomial p_t that move continuously to z^n:

$$p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_{n-1} z + a_n).$$

---

[2] This celebrated result was first proved by J. Wood (1798) and C.F.Gauss (1799), but with subtle gaps. A first correct proof was given by J-R. Argand in 1806. Nowadays, countless algebraic, topological, geometric and analytic proofs are available.

If we can define, for some r, loops

$$\tilde{f}_t(s) = \frac{p_t(r\exp(2\pi is))/p_t(r)}{|p_t(r\exp(2\pi is))/p_t(r)|},$$

then $\tilde{f}_1 = g_r$ and $\tilde{f}_0 = \exp(2\pi is) = \omega_n$. To make sure that we can construct such $\tilde{f}_t$, we have to make sure that none of the quantities we are dividing by can be 0, or in other words, that the polynomials $p_t(z)$ have no roots with $|z| = r$. We will show that this is the case if r is big enough. More specifically, let r be such that

$$r > \max\{|a_1| + \cdots + |a_n|, 1\}.$$

Then for $z \in \mathbb{C}$ with $|z| = r$ we have

$$\begin{aligned} |z|^n &> (|a_1| + \cdots + |a_n|)|z|^{n-1} \\ &> |a_1||z|^{n-1} + |a_2||z|^{n-2} + \cdots + |a_{n-1}||z| + |a_n| \\ &\geq |a_1 z^{n-1} + \cdots + a_n|. \end{aligned}$$

In particular, for $t \in [0, 1]$, the polynomials p_t cannot have a root with $|z| = r$, as the absolute value of $|z|^n$ is always bigger than that of the rest of the terms. It follows that the homotopy $\tilde{f}_t$ is well defined. $\square$

Lecture 7

Covering spaces are an essential tool in the derivation of the fundamental group of the circle, and also play an important role in algebraic topology and related fields. In this lecture we introduce and study covering spaces in some detail.

7.1 Covering spaces

Definition 7.1. A **covering** is a map $p\colon \tilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}$ of X, such that for every α, the preimage is a disjoint union of open sets

$$p^{-1}(U_\alpha) = \bigsqcup_\beta V_\alpha^\beta,$$

and such that the restriction $p|_{V_\alpha^\beta}\colon V_\alpha^\beta \to U_\alpha$ is a homeomorphism.

Example 7.2. For $k \in \mathbb{Z}$, the maps $p_k\colon S^1 \to S^1$, $z \mapsto z^k$ are covering maps. The preimage $p^{-1}(z)$ of any point $z = \exp(2\pi i t) \in S^1$ consists of precisely k distinct points, namely $\exp(2\pi i(t+j)/k)$ for $j \in \{0, \dots, k-1\}$. For $z = 1$, these are precisely the k-th complex roots of unity.

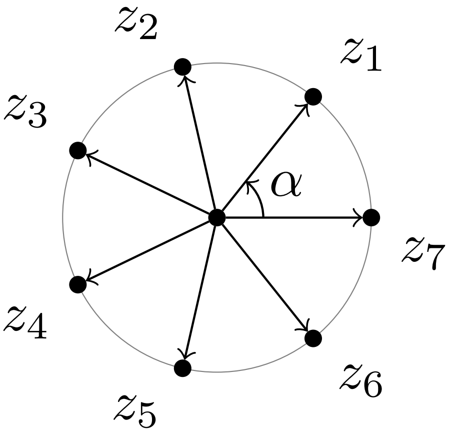


Figure 7.21: The preimage $p_7^{-1}(1)$.

Example 7.3. The map $p_\infty\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ is a covering map. The preimage $p_\infty^{-1}(z)$ consists of ∞ many points.

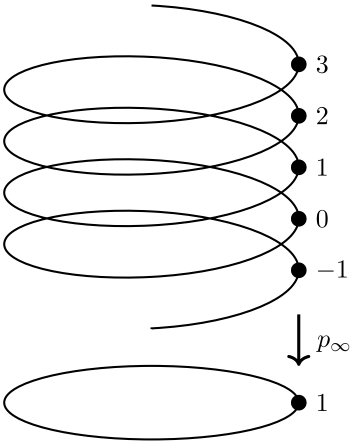


Figure 7.22: The preimage $p_\infty^{-1}(1)$.

Definition 7.4. A covering $p\colon \tilde{X} \to X$ is called an n**-fold covering** if for all $x \in X$, $p^{-1}(x)$ consists of precisely n points.

Definition 7.5. Two coverings $p\colon Y \to X$ and $q\colon Z \to X$ are called **isomorphic**, if there exists a homeomorphism $h\colon Y \to Z$ such that $p = q \circ h$.

It is common to visualize concepts such as the isomorphism of coverings via **commutative diagrams** such as the following.

$$\begin{tikzcd}
Y \arrow[rr, "h", "\cong"'] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\
& X &
\end{tikzcd}$$

The requirement is, that all compositions in such a diagram should coincide.

Example 7.6. The coverings $p_2\colon S^1 \to S^1$ and $p_{-2}\colon S^1 \to S^1$ are isomorphic: the homeomorphism $h\colon S^1 \to S^1$, $h(z) = 1/z$, satisfies $p_{-2} = p_2 \circ h$.

Example 7.7. The coverings $p_2\colon S^1 \to S^1$ and $p_3\colon S^1 \to S^1$ are not isomorphic: one is a 2-fold covering and the other is a 3-fold covering.

Definition 7.8. Let $p\colon \tilde{X} \to X$ be a covering. A **deck transformation** is a homeomorphism $\tau\colon \tilde{X} \to \tilde{X}$ such that $p \circ \tau = p$, i.e., τ gives rise to an isomorphism of a covering to itself. The set of all deck transformations of a cover is called $\mathrm{Deck}(p)$.

Exercise 7.9. Show that $(\mathrm{Deck}(p), \circ)$, where $\circ$ is the composition of maps, is a group.

Example 7.10. The map $\tau\colon S^1 \to S^1$, $z \mapsto -z$ gives a deck transformation for the cover $p_2\colon S^1 \to S^1$.

Exercise 7.11. Show that for $m \in \mathbb{Z}$, the maps $\tau_m\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto t + m$, give a deck transformation for the cover $p_\infty\colon \mathbb{R} \to S^1$. Conclude that $\mathrm{Deck}(p_\infty) \cong \mathbb{Z}$.

We next aim to construct a homomorphism from $\mathbb{Z}$ to the fundamental group $\pi_1(S^1, 1)$. To construct this homomorphism, we need to study how to lift homotopies from a space to a covering space.

Lecture 8

8.1 Liftings

Definition 8.1. Given a covering $p\colon \tilde{X} \to X$, a **lift** of $f\colon Y \to X$ was a map $\tilde{f}\colon Y \to \tilde{X}$ such that $f = p \circ \tilde{f}$,

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Example 8.2. Consider a loop $f\colon I \to S^1$, $t \mapsto \exp(2\pi i n t)$ and the covering p_∞. Then the map $\tilde{f}\colon I \to \mathbb{R}$, $t \mapsto nt$, is a lift of f.

Lemma 8.3. *Let $p\colon \tilde{X} \to X$ be a cover and $\tilde{f}, \tilde{g}\colon Y \to \tilde{X}$ maps. Then:*

(1) $\tilde{f}$ is a lift of $p \circ \tilde{f}$;

(2) If $\tilde{f} \simeq \tilde{g}$, then $p \circ \tilde{f} \simeq p \circ \tilde{g}$ ("Homotopies descend");

(3) If $\alpha, \beta\colon I \to \tilde{X}$ are paths with $\alpha(1) = \beta(0)$, then $p \circ (\alpha * \beta) = (p \circ \alpha) * (p \circ \beta)$ ("Paths descend").

Proof. Property (1) is obvious from the definition of a lift. For property (2), observe that any homotopy $\tilde{f}_t$ from $\tilde{f}$ to $\tilde{g}$ gives rise to a homotopy $p \circ \tilde{f}_t$ from $p \circ \tilde{f}$ to $p \circ \tilde{g}$. For property (3), note that

$$p \circ (\alpha * \beta)(t) = \begin{cases} p \circ \alpha(2t) & t \leq 1/2 \\ p \circ \beta(2t-1) & t \geq 1/2 \end{cases},$$

which is the same as $(p \circ \alpha) * (p \circ \beta)(t)$. $\square$

8.2 Loops on S^1

Recall the map

$$\begin{aligned} \omega_n\colon I &\to S^1 \\ t &\mapsto \exp(2\pi i n t). \end{aligned}$$

The map $\tilde{\omega}_n \colon I \to \mathbb{R}$, $t \colon nt$ is clearly a lift of ω_n, i.e., it satisfies

$$\omega_n = p \circ \tilde{\omega}_n. \tag{A}$$

Consider also the deck transformation $\tau_n \colon \mathbb{R} \to \mathbb{R}$, $t \mapsto t+n$, and the composition $\tilde{\omega}_m * (\tau_m \circ \tilde{\omega}_n)$. This composition is a path in $\mathbb{R}$ from 0 to $m+n$, and therefore homotopic to $\tilde{\omega}_{m+n}$,

$$\tilde{\omega}_{n+m} \simeq \tilde{\omega}_m * (\tau_m \circ \tilde{\omega}_n), \tag{B}$$

as can be seen using the straight-line homotopy $f_t = (1-t)\tilde{\omega}_{n+m} + t\tilde{\omega}_m * (\tau_m \circ \tilde{\omega}_n)$.

We now have everything in place to construct a homomorphism of $\mathbb{Z}$ to the fundamental group of the circle. Define the map

$$\begin{aligned} \Phi \colon \mathbb{Z} &\to \pi_1(S^1, 1) \\ n &\mapsto [\omega_n]. \end{aligned}$$

Proposition 8.4. *The map $\Phi \colon \mathbb{Z} \to \pi_1(S^1, 1)$, $\Phi(n) = [\omega_n]$, is a group homomorphism.*

Proof. We need to show that $[\omega_{m+n}] = [\omega_m] \bullet [\omega_n]$:

$$\begin{aligned} \Phi(m+n) &= [\omega_{m+n}] \\ &\overset{\text{(A)}}{=} [p \circ \tilde{\omega}_{m+n}] \\ &\overset{\text{Lemma (8.3)(2)+(B)}}{=} [p \circ (\tilde{\omega}_m * (\tau_m \circ \tilde{\omega}_n))] \\ &\overset{\text{Lemma (8.3)(3)}}{=} [p \circ \tilde{\omega}_m * p \circ \tau_m \circ \tilde{\omega}_n] \\ &= [p \circ \tilde{\omega}_m] \bullet [p \circ \tau_m \circ \tilde{\omega}_n] \\ &\overset{\tau_m \in \text{Deck}(p)}{=} [p \circ \tilde{\omega}_m] \bullet [p \circ \tilde{\omega}_n] \\ &\overset{\text{(A)}}{=} [\omega_m] \bullet [\omega_n] \\ &= \Phi(m) \bullet \Phi(n). \end{aligned}$$

□

The philosophy of the proof is that we have shown something about loops on S^1 by considering a cover of S^1, $p_\infty \colon \mathbb{R} \to S^1$, and working in $\mathbb{R}$. Things are very simple in $\mathbb{R}$: the crucial property (B) is easy to prove and shows that the composition of two lifts $\tilde{\omega}_m$ and $\tilde{\omega}_n$ (up to a reparametrization given by the deck transformation τ_m) is homotopic to the lift $\tilde{\omega}_{m+n}$. Using the property that “homotopies descend” and “paths descend”, we can transfer things proved “upstairs” to “downstairs”.

What the proof does not show yet, is that the homomorphism Φ is bijective: we don’t know whether Φ hits all the elements of $\pi_1(S^1, 1)$, and whether two distinct $n \neq m$ give rise to distinct classes $[\omega_n]$ and $[\omega_m]$. The latter is equivalent to the important statement that for all $m \in \mathbb{Z}$, $\omega_m \simeq e \Leftrightarrow m = 0$ (where e is the constant loop at 1). This statement is non-trivial, and relies on the fact that homotopies in the base space of a covering “lift” to homotopies in the covering space.

8.3 The homotopy lifting property

Recall the convention that for a homotopy $F\colon Y \times I \to X$ we write $f_t(y) = F(y, t)$.

Definition 8.5. Let $p\colon Z \to X$ be a map. Then p has the **Homotopy Lifting Property** (HLP) if given a homotopy $F\colon Y \times I \to X$ and a *lift* $g\colon Y \times \{0\} \to Z$ of f_0, so that $f_0 = p \circ g$, there exists a *unique* homotopy $\tilde{F}\colon Y \times I \to Z$ such that

(i) $\tilde{f}_0 = g$;

(ii) $p \circ \tilde{F} = F$.

In terms of diagrams,

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "g"] \arrow[d, hook, "\iota"] & Z \arrow[d, "p"] \\
Y \times I \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & X
\end{tikzcd}$$

Recall that we use the notation $Y \hookrightarrow X$ to denote the *inclusion map* of a subspace. The diagram is required to commute, i.e., all compositions coincide (for example, $p \circ g = F \circ \iota$). The dashed line means that we require the existence of a map $\tilde{F}$ making the diagram commute. Note that Condition (i) above says that the upper triangle in the diagram commutes ($\tilde{f}_0 = \tilde{F} \circ \iota = g_0$) and Condition (ii) is equivalent to the commutativity of the lower triangle.

An important special case is the **Path Lifting Property**, or homotopy lifting property for paths.

Definition 8.6. Let $p\colon Z \to X$ be a map. Then p satisfies the homotopy lifting property for paths, or the Path Lifting Property (PLP), if for any path $f\colon I \to X$ with $f(0) = x_0$ and $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique path $\tilde{f}\colon I \to Z$ with $\tilde{f}(0) = \tilde{x}_0$ and $p \circ \tilde{f} = f$.

Note that the path lifting property is a special case of the HLP with $Y = \{\mathrm{pt}\}$. In this case, the homotopy F is simply a path

$$F\colon \{\mathrm{pt}\} \times I \to X,$$

$f_0\colon \{\mathrm{pt}\} \times \{0\} \to X$ is simply a point $x_0 \in X$, and $g\colon \{\mathrm{pt}\} \times \{0\}$ is simply a point $\tilde{x}_0 \in Z$. Denoting $f(t) = F(\mathrm{pt}, t)$, we recover Definition 8.6.

Lecture 9

In this lecture we use the Homotopy Lifting Property to prove that the fundamental group of the circle is isomorphic to $\mathbb{Z}$. We will use the following result about coverings.

Proposition 9.1. *Covering maps satisfy the homotopy lifting property.*

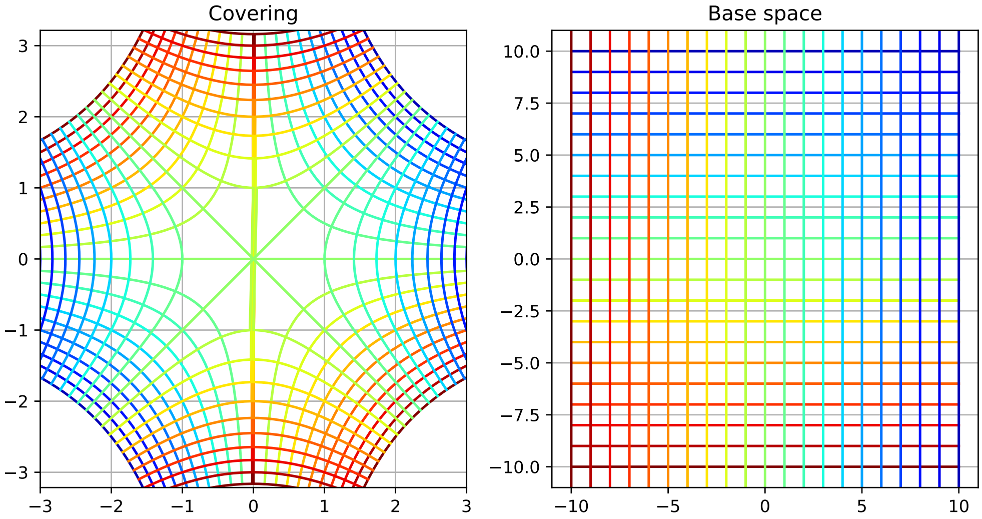


Figure 9.23: The covering map $\mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$, $z \mapsto z^2$ with the liftings of two paths (a vertical and a horizontal line).

9.1 The fundamental group of the circle

Theorem 9.2. *The map* $\Phi\colon \mathbb{Z} \to \pi_1(S^1, 1)$, $n \mapsto [\omega_n]$, *is a group isomorphism.*

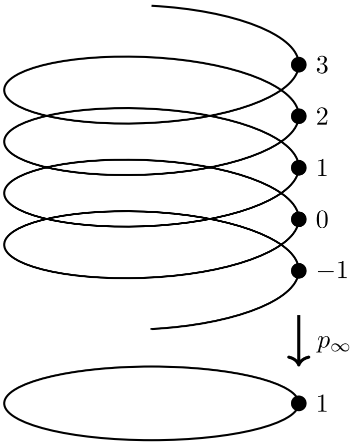
Proof. We already saw that Φ is a homomorphism, and only need to show that it is bijective.

We first show that the map is surjective: if $[\alpha] \in \pi_1(S^1, 1)$ then there exists $n \in \mathbb{Z}$ with $[\alpha] = [\omega_n]$. Consider again the cover $p = p_\infty\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$. Since the covering p satisfies the HLP by Proposition 9.1, and hence also the PLP, given a loop α there exists a unique lift $\tilde{\alpha}\colon I \to \mathbb{R}$ such that

(i) $p \circ \tilde{\alpha} = \alpha$;

(ii) $\tilde{\alpha}(0) = 0$.

Since $\alpha(1) = 1$ (α is a loop starting and ending at $1 \in S^1$) and $p \circ \tilde{\alpha} = \alpha$, we have $\tilde{\alpha}(1) \in p^{-1}(1) = \mathbb{Z}$, say $\tilde{\alpha}(1) = n$.



Therefore $\tilde{\alpha} \overset{\partial}{\simeq} \tilde{\omega}_n$, since both are paths from 0 to n in $\mathbb{R}$, with a homotopy given by the straight-line homotopy $f_t = (1-t)\tilde{\alpha} + t\tilde{\omega}_n$. Since homotopies descend, we get

$$\alpha = p \circ \tilde{\alpha} \overset{\partial}{\simeq} p \circ \tilde{\omega}_n = \omega_n,$$

which implies $[\alpha] = [\omega_n]$.

To show injectivity, assume that $\Phi(n) = [\omega_n] = [e]$, i.e., $\omega_n \overset{\partial}{\simeq} e$, the constant loop. This means that there is a homotopy of loops

$$F \colon I \times I \to S^1$$

with $f_0 = \omega_n$, $f_1 = e$, and $f_t(0) = f_t(1) = 1$ for all t. Define $g \colon I \times \{0\} \to \mathbb{R}$ by setting $g(s,0) = \tilde{\omega}_n$. By Proposition 9.1, the covering p satisfies the HLP, and we therefore have a homotopy $\tilde{F} \colon I \times I \to \mathbb{R}$ such that $\tilde{f}_0 = g$ and $p \circ \tilde{F} = F$. The other end of the homotopy, $\tilde{f}_1$, satisfies $p \circ \tilde{f}_1 = e$, the constant loop. Therefore:

- $\tilde{f}_0(0) = 0$ since $\tilde{f}_0 = \tilde{\omega}_n$;
- $\tilde{f}_t(0) \in \mathbb{Z}$ since $p \circ \tilde{f}_t(0) = f_t(0) = 1$;
- $\tilde{f}_1(s) \in \mathbb{Z}$ since $p \circ \tilde{f}_1(s) = e(s) = 1$;
- $\tilde{f}_t(1) \in \mathbb{Z}$ since $p \circ \tilde{f}_t(1) = f_t(1) = 1$;
- $\tilde{f}_0(1) = n$ since $\tilde{f}_0 = \tilde{\omega}_n$.

Since we consider $\mathbb{R}$ with the Euclidean (metric) topology, a continuous map that only takes values in $\mathbb{Z}$ is constant (the continuous image of a connected topological space is connected). Therefore,

$$0 = \tilde{f}_0(0) = \tilde{f}_t(0) = \tilde{f}_1(s) = \tilde{f}_t(1) = \tilde{f}_0(1) = n.$$

This completes the proof. $\square$

Lecture 10

In this lecture we will complete the last missing piece in the derivation of the fundamental group $\pi_1(S^1, 1)$. We restate the result.

Proposition 10.1. *Covering maps satisfy the homotopy lifting property.*

10.1 The local Homotopy Lifting Property

Consider the special case where $p\colon \tilde{X} \to X$ is a covering such that $\tilde{X} = \bigsqcup_\beta V^\beta$, with each $V^\beta \cong X$. Let $F\colon Y \times I \to X$ and a lift $g\colon Y \times \{0\} \to \tilde{X}$ such that $f_0 = p \circ g$. If $g(Y \times \{0\}) \subset V^\beta$ for some β, then we can lift the homotopy F to a homotopy $\tilde{F}$ that extends g by simply applying the homeomorphism $q^\beta\colon X \to V^\beta$ to F that is inverse to $p|_{V^\beta}$. In general, however, we can only do this construction "locally", that is, within an open set U_α, and need to make sure that it can be extended.

Lemma 10.2. *Let $p\colon \tilde{X} \to X$ be a covering and let $F\colon Y \times I \to X$ be a homotopy. Let $g\colon Y \times \{0\} \to \tilde{X}$ be such that $p \circ g = f_0$. Then for every $y_0 \in Y$ there exists an open set N with $y_0 \in N \subset Y$ and a* unique *homotopy (depending on N)*

$$\tilde{F}_N\colon N \times I \to \tilde{X}$$

such that $p \circ \tilde{F}_N = F|_{N \times I}$ and $(\tilde{f}_N)_0 = g|_{N \times \{0\}}$. Moreover, if M is another such neighbourhood, with $y_0 \in M \subset Y$, then

$$\tilde{F}_M|_{(M \cap N) \times I} = \tilde{F}_N|_{(M \cap N) \times I} = \tilde{F}_{M \cap N}. \tag{10.1}$$

For the proof we require a special version of the Lebesgue Covering Lemma.

Lemma 10.3. *Let $I = \bigcup_\alpha I_\alpha$ an open cover. Then there exists $\varepsilon > 0$ such that for every $A \subset I$ with $\operatorname{diam}(A) < \varepsilon$, $A \subset I_\alpha$ for some α.*

Proof of Lemma 10.2. Let $p\colon \tilde{X} \to X$ be a covering map. Assume we have a homotopy $F\colon Y \times I \to X$ and "initial data" $g\colon Y \to \tilde{X}$, so that $p \circ g = f_0$. By the definition of a covering, we have an open cover $\{U_\alpha\}$ of X, and for each α a collection of disjoint subsets $\{V_\alpha^\beta\}$ of $\tilde{X}$ such that $p^{-1}(U_\alpha) = \bigsqcup V_\alpha^\beta$, and the restriction $p|_{V_\alpha^\beta}\colon V_\alpha^\beta \to U_\alpha$ is a homeomorphism. For every pair (α, β), denote by $q_{\alpha,\beta}\colon U_\alpha \to V_\alpha^\beta$ the inverse of this homeomorphism. Since F is continuous, for every $(y, t) \in Y \times I$ there exists an open neighbourhood $N \times (a, b) \subset Y \times I$

and an index α such that $F(y,t) \in U_\alpha$ for $(y,t) \in N \times (a,b)$. For every fixed y and as t ranges over I, we get various subsets with this property, and since I is compact, there are finitely many such $N_i \times I_i$ covering $\{y\} \times I$. Set $N = \cap_i N_i$. By Lemma 10.3 we can choose a sufficiently fine partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that every interval (t_j, t_{j+1}) is contained in one of the I_i. Therefore, every i there is an α with $F(N \times (t_i, t_{i+1})) \subset U_\alpha$.

We now claim that there is a sequence of maps $\tilde{F}_N^k$ such that

1. $\tilde{F}_N^k \colon N \times [0, t_k] \to \tilde{X}$ is a lift of $F|_{N \times [0,t_k]}$;
2. $(\tilde{F}_N^k)_0 = g|_N$;
3. $\tilde{F}_N^{k+1}|_{N \times [0,t_k]} = \tilde{F}_N^k$.

Moreover, these three properties determine the sequence $\{\tilde{F}_N^k\}$ uniquely. We then set $F_N = F_N^n$. We construct the sequence of maps $\tilde{F}_N^k$ by induction. Clearly, there is only one way to define $\tilde{F}_N^0$ on $N \times [0,0]$ such that $(\tilde{F}_N^0)_0 = g|_N$. Assume now that we have a sequence of maps $\tilde{F}_N^j$ up to $j = k$. By assumption, there is an index α such that $F(N \times [t_k, t_{k+1}]) \subset U_\alpha$. By making N smaller, if necessary, we can assume that $\tilde{F}_N^k|_{N \times \{t_k\}} \subset V_\alpha^\beta$ for some β, and that if we define

$$\tilde{E} = q_{\alpha,\beta} \circ F|_{N \times [t_k, t_{k+1}]},$$

then

$$\tilde{E}|_{N \times \{t_k\}} = \tilde{F}_N^k|_{N \times \{t_k\}}.$$

Now define the extension

$$\tilde{F}_N^{k+1}(z,t) = \begin{cases} \tilde{F}_N^k(z,t), & t \leq t_k \\ \tilde{E}(z,t) & t \in [t_k, t_{k+1}]. \end{cases}$$

By the Pasting Lemma, $\tilde{F}_N^{k+1}$ is continuous. By construction, the resulting map satisfies conditions (1)-(3) above.

Assume now that we have two maps, $\tilde{F}_N$, $\tilde{F}'_N$, constructed in this fashion. It is enough to show that, for any $z \in N$, $\tilde{F}_N|_{\{z\} \times I} = \tilde{F}'_N|_{\{z\} \times I}$. As before, let $0 = t_0 < t_1 < \cdots < t_m = 1$ be a partition such that $F(\{z\} \times [t_j, t_{j+1}]) \subset U_\alpha$. We proceed by induction. It is clear that both maps have to coincide on $\{z\} \times [0,0]$, as both have to match $g(z,0)$ there. Assume that $\tilde{F}'_N = \tilde{F}_N$ on $[0, t_k]$. Since $[t_k, t_{k+1}]$ is connected, there exists a unique β such that $\tilde{F}_N(\{z\} \times [t_k, t_{k+1}])$ is contained in V_α^β. Similarly, there is a unique β' such that $\tilde{F}'_N(\{z\} \times [t_k, t_{k+1}])$ is in $V_\alpha^{\beta'}$. But since $\tilde{F}_N(z, t_k) = \tilde{F}'_N(z, t_k)$, we have to have $\beta = \beta'$. By construction of the extension $\tilde{E}$, the two maps also coincide on $\{z\} \times [0, t_{k+1}]$.

The proof also shows that if we take two neighbourhoods N, M with the properties just derived, then by uniqueness we have $\tilde{F}_M|_{(M \cap N) \times I} = \tilde{F}_N|_{(M \cap N) \times I}$. $\square$

Proof of Proposition 10.1. Cover $Y \times I$ with open sets $N \times I$, as guaranteed by Lemma 10.2. We then get a family of lifts $\tilde{F}_N \colon N \times I \to \tilde{X}$ that coincide on the intersection of two sets in the cover. Hence, by the Pasting Lemma, they are continuous and therefore lift F. $\square$

Lecture 11

We computed the fundamental group of some elementary spaces, but haven't really seen what this means yet. For example, if we denote the closed disk in $\mathbb{C} \cong \mathbb{R}^2$ by

$$\mathbb{D}^2 := \{z \in \mathbb{C} \mid |z| \leq 1\},$$

then

$$\pi_1(\mathbb{D}^2, 1) = \{0\}, \quad \pi_1(S^1, 1) \cong \mathbb{Z}.$$

What does this say about the underlying topological spaces? As we will see, this implies (for example) that S^1 cannot be a retract of $\mathbb{D}^2$, which in turn has other consequences such as the Brouwer Fixed Point Theorem (a map $\mathbb{D}^2 \to \mathbb{D}^2$ has a fixed-point). Ultimately, we would like to show that the fundamental group is a **homotopy invariant**: homotopy equivalent spaces have the same fundamental group. We first need to study how the fundamental group reacts to continuous maps between spaces.

11.1 Induced homomorphisms

Recall that a pair of spaces is a pair of topological spaces (X, A) with $A \subset X$.

Definition 11.1. A **map of pairs**

$$f \colon (X, A) \to (Y, B)$$

is a map $f \colon X \to Y$ such that $f(A) \subset B$.

Example 11.2. The typical example is when $A = \{x_0\}$ and $B = \{y_0\}$, in which case we write $f \colon (X, x_0) \to (Y, y_0)$ to denote a map with $f(x_0) = y_0$.

Example 11.3. Consider the two-fold cover $f \colon (S^1, 1) \to (S^1, 1)$, $z \mapsto z^2$.

Definition 11.4. The **induced homomorphism** of $f \colon (X, x_0) \to (Y, y_0)$ is the map

$$\begin{aligned} f_* \colon \pi_1(X, x_0) &\to \pi_1(Y, y_0) \\ [\alpha] &\mapsto [f \circ \alpha]. \end{aligned}$$

The induced homomorphism is also sometimes called a *push-forward*.

Lemma 11.5. *The map f_* is a group homomorphism.*

Proof. We first have to verify that this is a well-defined map, i.e., that if $\alpha \overset{\partial}{\simeq} \beta$ then $f \circ \alpha \overset{\partial}{\simeq} f \circ \beta$. This is clear: if $F \colon I \times I \to X$ is a homotopy with $f_0 = \alpha$ and $f_1 = \beta$, then $G = f \circ F$ is a homotopy from $f \circ \alpha$ to $f \circ \beta$. To show that f_* is a homomorphism, we need to show that $f_*([\alpha] \bullet [\beta]) = f_*([\alpha]) \bullet f_*([\beta])$. Since

$$f_*([\alpha] \bullet [\beta]) = f_*([\alpha * \beta]) = [f \circ (\alpha * \beta)]$$

and

$$f_*([\alpha]) \bullet f_*([\beta]) = [f \circ \alpha] \bullet [f \circ \beta] = [(f \circ \alpha) * (f \circ \beta)],$$

we are left with showing that $f \circ (\alpha * \beta) \overset{\partial}{\simeq} (f \circ \alpha) * (f \circ \beta)$. Note that, by definition of the concatenation of paths,

$$(f \circ \alpha) * (g \circ \beta) = \begin{cases} f \circ \alpha(2s) & s \leq 1/2 \\ f \circ \beta(2s-1) & s \geq 1/2 \end{cases}.$$

which is the same as the definition of $f \circ (\alpha * \beta)$. This completes the proof. ☐

Example 11.6. Consider the covering map $p_2 \colon (S^1, 1) \to (S^1, 1)$, $z \mapsto z^2$. Let $\omega_n \colon I \to S^1$ be the map $\omega_n(s) = \exp(2\pi i n s)$. Then $p_2 \circ \omega_n = \omega_{2n}$, and $(p_2)_*([\omega_n]) = ([\omega_{2n}])$. The induced map on $\mathbb{Z}$ is the doubling map

$$\begin{tikzcd} \pi_1(S^1,1) \arrow[r, "(p_2)_*"] \arrow[d, "\cong"] & \pi_1(S^1,1) \arrow[d, "\cong"] \\ \mathbb{Z} \arrow[r, "n \mapsto 2n"] & \mathbb{Z} \end{tikzcd}$$

One similarly derives the induced map for $z \mapsto z^d$.

The next lemma shows that the fundamental group is a *functor*.

Lemma 11.7. *The induced homomorphism satisfies the properties:*

1. $(\mathrm{Id}_{(X,x_0)})_* = \mathrm{Id}_{\pi_1(X,x_0)}$*;*
2. *If* $f \colon (X, x_0) \to (Y, y_0)$ *and* $g \colon (Y, y_0) \to (Z, z_0)$*, then*

$$(g \circ f)_* = g_* \circ f_*$$

Proof. The first property is obvious: if nothing happens at the topological level, then nothing can happen at the algebraic level. For the second property, note that

$$(g \circ f)_*([\gamma]) = [g \circ f \circ \gamma] = g_*([f \circ g]) = (g_* \circ f_*)([\gamma]).$$

☐

An immediate consequence is that the fundamental group maps homeomorphic spaces to isomorphic groups. This allows to distinguish spaces: if two spaces X and Y have different fundamental group, they cannot be homeomorphic.

Theorem 11.8. *If $f\colon (X,x_0)\to(Y,y_0)$ is a homeomorphism, then $f_*\colon \pi_1(X,x_0)\to\pi_1(Y,y_0)$ is a group isomorphism.*

Proof. We apply Lemma 11.7(1) and (2) with $g=f^{-1}$. Then

$$\mathrm{Id}_{\pi_1(X,x_0)} = (\mathrm{Id}_{(X,x_0)})_* = (f^{-1}\circ f)_* = (f^{-1})_*\circ f_*,$$

and similarly (reversing the role of f^{-1} and f) $\mathrm{Id}_{\pi_1(X,x_0)} = f_*\circ(f^{-1})_*$, which shows that $(f^{-1})_* = f_*^{-1}$. □

Example 11.9. Since $\pi_1(\mathbb{D}^2,1)=\{0\}$ and $\pi_1(S^1,1)\cong\mathbb{Z}$, $(\mathbb{D}^2,1)\ncong(S^1,1)$.

In the following lectures we will see that this extends to *homotopy equivalence*.

11.2 Categories and functors

A **category** $\mathcal{C}$ consists of objects $\mathrm{obj}(\mathcal{C})$, for any ordered pair of objects (a,b) a set $\mathrm{Hom}_{\mathcal{C}}(a,b)$ whose elements are called morphisms or arrows (often written, $a\xrightarrow{f}b$) and composition maps $\mathrm{Hom}_{\mathcal{C}}(a,b)\times\mathrm{Hom}_{\mathcal{C}}(b,c)\to\mathrm{Hom}_{\mathcal{C}}(a,c)$, $(f,g)\mapsto g\circ f$, such that

1. (associativity) if $f\in\mathrm{Hom}_{\mathcal{C}}(a,b)$, $g\in\mathrm{Hom}_{\mathcal{C}}(b,c)$ and $h\in\mathrm{Hom}_{\mathcal{C}}(c,d)$, then $h\circ(g\circ f)=(h\circ g)\circ f$;
2. (identity) for every $a\in\mathrm{obj}(\mathcal{C})$ there exists $\mathrm{id}_a\in\mathrm{Hom}_{\mathcal{C}}(a,a)$ such that $f\circ\mathrm{id}_a=\mathrm{id}_b\circ f=f$ for any $f\in\mathrm{Hom}_{\mathcal{C}}(a,b)$.

In applications, the objects are often sets with a certain structures (vector spaces, topological spaces, groups) and the morphisms are structure-preserving maps between them (linear maps, continuous functions, group homomorphisms). While in these examples the objects are denoted by V, X, or G, the lower-case notation for objects in an arbitrary category indicates that there is no a priori requirement for these to be sets.

Let $\mathcal{C}$, $\mathcal{D}$ be two categories. A **functor** $F\colon\mathcal{C}\to\mathcal{D}$ assign to every object $a\in\mathrm{obj}(\mathcal{C})$ an object $F(a)\in\mathrm{obj}(\mathcal{D})$, and to every morphism $f\in\mathrm{Hom}_{\mathcal{C}}(a,b)$ a morphism $F(f)\in\mathrm{Hom}_{\mathcal{D}}(F(a),F(b))$, in such a way that

1. $F(\mathrm{id}_a)=\mathrm{id}_{F(a)}$;
2. if $f\in\mathrm{Hom}_{\mathcal{C}}(a,b)$ and $g\in\mathrm{Hom}_{\mathcal{C}}(b,c)$, then $F(g\circ f)=F(g)\circ F(f)$.

Example 11.10. Let Top_0 denote the category whose objects are pointed topological spaces (X,x_0), and whose morphisms are maps of pairs $(X,x_0)\xrightarrow{f}(Y,y_0)$. Let $\mathcal{G}$ be the category of groups, whose morphisms are group homomorphisms. Then the fundamental group π_1 is a functor:

$$\pi_1\colon \mathrm{Top}_0\to\mathcal{G}.$$

1. Every object (X,x_0) is assigned to a group $\pi_1(X,x_0)$;

2. Any map $(X, x_0) \xrightarrow{f} (Y, y_0)$ gives rise to a group homomorphism $\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0)$, where we write $f_* = \pi_1(f)$;
3. The identity gets mapped to the identity: $(\mathrm{Id}_{(X,x_0)})_* = \mathrm{Id}_{\pi_1(X,x_0)}$;
4. We have the property that $(g \circ f)_* = g_* \circ f_*$.

A functor as defined here is also called a *covariant* functor, because it preserves the direction of arrows. A *contravariant* functor is one that reverses the direction.

The language of categories and functors, sometimes also called “abstract nonsense”, forms the basis of the modern treatment of many fields of mathematics, including algebraic geometry, number theory, and algebraic topology. It allows the study of structural similarities between mathematics concepts in an elegant way, and in particular it allows to transfer topological ideas to other fields of mathematics.

Lecture 12

In the previous lecture we saw that the fundamental group is a functor: maps between pointed topological spaces get assigned to group homomorphisms in a way that preserves the identity map and compositions. We also saw that homeomorphisms correspond to isomorphisms in the category of groups. In this lecture we will study the effect of retractions, and deformation retracts, on the fundamental group.

12.1 Retractions

Let (X, A) be a pair of topological spaces, witht $A \subset X$. Recall that a **retraction** is a map $r\colon X \to A$ such that $r|_A = \mathrm{Id}_A$.

Example 12.1. The map $\mathbb{C} - \{0\} \to S^1$, $z \mapsto |z|$, is a retraction.

A kind of converse to a retraction is the **inclusion** $\iota\colon A \hookrightarrow X$. We have the composition $r \circ \iota = \mathrm{Id}_A$, and the reverse composition $\iota \circ r\colon X \to X$. In diagrams,

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{Id}_A"] \arrow[dr, hook, "\iota"] & & A \arrow[dr, hook, "\iota"] & \\
& X \arrow[ur, two heads, "r"] \arrow[rr, "\iota\circ r"] & & X
\end{tikzcd}$$

where the arrow with hook $\hookrightarrow$ is used to emphasize that the map is *injective*, while the arrow with two tips $\twoheadrightarrow$ is used to emphasize that the map is *surjective*, or onto. A retract r is called a **deformation retract**, if

$$\iota \circ r \overset{A}{\simeq} \mathrm{Id}_X,$$

which means that there is a homotopy from $\iota \circ r$ to the identity Id_X that does nothing on A.

Proposition 12.2. *Let $A \subset X$, $r\colon X \to A$ a retraction, and $\iota\colon A \to X$ the inclusion. Let $x_0 \in A$ and let $r_*\colon \pi_1(X, x_0) \to \pi_1(A, x_0)$ and $\iota_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ be the induced (push-forward) maps between the fundamental groups. Then:*

1. *r_* is surjective and ι_* is injective;*

2. *If r is a* deformation retract*, then r_* and ι_* are isomorphisms.*

Note that, in particular, for a deformation retract r, the map

$$(\iota \circ r)_*\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$$

is an isomorphism (though it may not be the identity, as is the case with $(r \circ \iota)_*$).

Proof. The first claim is clear: since $\mathrm{Id}_{\pi_1(A,x_0)} = (r \circ \iota)_* = r_* \circ \iota_*$, ι_* has to be injective (otherwise the composition couldn't be injective) and r_* has to be surjective (otherwise the composition couldn't be surjective).

To show that r_* is an isomorphism if r is a deformation retract, it is enough to show that r_* is injective. Let $[\gamma] \in \pi_1(X, x_0)$ and assume that $r_*([\gamma]) = [r \circ \gamma] = [e_A]$, or equivalently $r \circ \gamma \overset{\partial}{\simeq} e_A$, where e_A is the constant loop at x_0 *in* A. We need to show that in this case, $[\gamma] = [e_X]$, or $\gamma \overset{\partial}{\simeq} e_X$, where e_X is the constant loop at x_0 *in* X.

As $r \circ \gamma$ is a loop in $A \subset X$, $\iota \circ r \circ \gamma$ it is a loop in X, and $\iota \circ r \circ \gamma \overset{\partial}{\simeq} e_X$ by the same homotopy that takes $r \circ \gamma$ to e_A. By the transitivity of homotopy, it is therefore enough to show that

$$\iota \circ r \circ \gamma \overset{\partial}{\simeq} \gamma.$$

This follows by simply applying the homotopy from $\iota \circ r$ to Id_X to the loop γ. More precisely, let $F\colon X \times I \to X$ be the homotopy from $\iota \circ r$ to Id_X, so that $f_0 = \iota \circ r$ and $f_1 = \mathrm{Id}_X$. Construct a new homotopy $G\colon I \times I \to X$ by setting $G(s,t) = F(\gamma(s), t)$. Then $g_0 = \iota \circ r \circ \gamma$ and $g_1 = \gamma$. This concludes the proof. ☐

12.2 Homotopy invariance

So far we have seen that deformation retracts give rise to isomorphic fundamental groups. We next show that this holds more generally for homotopy equivalence.

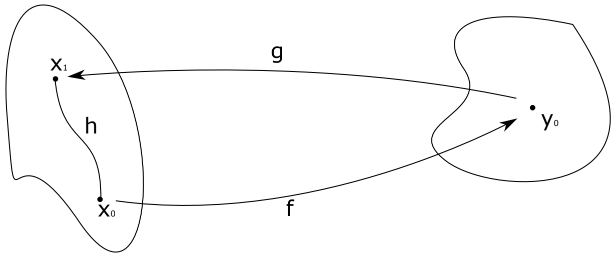
Proposition 12.3. *Let $f\colon X \to Y$ be a homotopy equivalence. Then for any $x_0 \in X$, the induced map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

Proof. Let $g\colon Y \to X$ be a homotopy inverse, so that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. Set $y_0 = f(x_0)$ and $x_1 = g(y_0)$. The composition $g \circ f$ thus gives rise to a homomorphism of fundamental groups

$$\begin{aligned}(g \circ f)_*\colon \pi_1(X, x_0) &\to \pi_1(X, x_1)\\ [\gamma] &\mapsto [g \circ f \circ \gamma].\end{aligned}$$

Let $K\colon X \times I \to X$ be a homotopy from Id_X to $g \circ f$, and define a path $h\colon I \to X$ from x_0 to x_1 by $h(t) = K(x_0, t)$. By Proposition 6.1 (Lecture 6), this path induces an isomorphism of fundamental groups

$$\begin{aligned}\beta_h\colon \pi_1(X, x_0) &\to \pi_1(X, x_1)\\ [\gamma] &\mapsto [\overline{h} * \gamma * h],\end{aligned}$$



where $\overline{h}(t) = h(1-t)$ is the inverse path, from x_1 to x_0. We claim that $\beta_h = (g \circ f)_*$. To show that these homomorphisms coincide, for any $[\gamma] \in \pi_1(X, x_0)$ we will construct a homotopy between $\overline{h} * \gamma * h$ and $g \circ f \circ \gamma$.

Define first a homotopy

$$h_t(s) = H(t,s) = \begin{cases} h(s) & \text{if } s \geq t \\ h(t) & \text{if } s \leq t \end{cases},$$

so that $h_1(s) = h(1) = x_1$ and $h_0(s) = h(s)$. In addition, define the homotopy

$$\gamma_t(s) = K(\gamma(s), t),$$

which consists in applying the homotopy from Id_X to $g \circ f$ to the loop γ. In particular, $\gamma_0 = \gamma$ and $\gamma_1 = g \circ f \circ \gamma$. Finally, consider the homotopy

$$\alpha_t(s) = \overline{h}_t * \gamma_t * h_t(s).$$

One checks directly from the definition that the endpoints of each of the concatenated paths coincide, and that $\alpha_0 = \overline{h} * \gamma * h$ and $\alpha_1 = g \circ f \circ \gamma$. We therefore have a homotopy $\overline{h} * \gamma * h \overset{\partial}{\simeq} g \circ f \circ \gamma$ and hence

$$\beta_h([\gamma]) = [\overline{h} * \gamma * h] = [g \circ f \circ \gamma] = (g \circ f)_*([\gamma]).$$

In particular, $(g \circ f)_* = g_* \circ f_*$ is an isomorphism, and therefore f_* is injective and g_* is surjective. Repeating the proof in the other direction (roles of f and g reversed), shows that f_* is surjetive and g_* is injective, thus finishing the proof that we have an isomorphism. $\square$

Lecture 13

The first obvious application of the fact that retractions induce surjective homomorphisms in the fundamental group is the following.

Proposition 13.1. *There is not retract* $\mathbb{D}^2 \to S^1$.

Proof. The existence of a retraction would imply a surjection $\pi_1(\mathbb{D}^2, 1) \twoheadrightarrow \pi_1(S^1, 1)$, but the fundamental group of the disk is $\pi_1(\mathbb{D}^1, 1) = \{0\}$, and the fundamental group of the circle is $\pi_1(S^1, 1) \cong \mathbb{Z}$. ☐

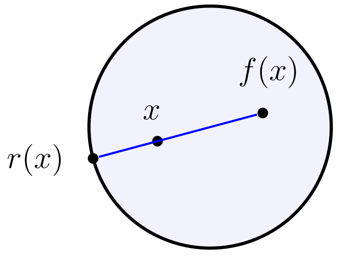
An important consequence of this "no retract theorem" is the celebrated Brouwer Fixed Point Theorem.

13.1 The Brouwer Fixed Point Theorem

The following important result generalizes the fact that a continuous function $f\colon I \to I$ has a fixed point, i.e., a point $x \in I$ such that $f(x) = x$. In the case of the interval, this is an easy consequence of the intermediate value theorem. The generalization to maps from $I \times I$ to itself, or equivalently, from $\mathbb{D}^2$ to itself, is surprisingly non-trivial. A proof was found by Luitzen Egbertus Jan (Bertus) Brouwer in 1910.

Theorem 13.2. *(Brouwer Fixed Point Theorem) Every map* $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ *has a fixed point.*

Proof. The proof is by contradiction. Assume that the statement is wrong, and that there is a map $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ such that $f(x) \neq x$ for all $x \in \mathbb{D}^2$. For every $x \in \mathbb{D}^2$, there is a unique line joining through x and $f(x)$, parametrized by $L_x(t) = (1-t)f(x) + tx$ for $t \in \mathbb{R}$. This line intersects the boundary circle S^1 in exactly in two points, one for which $t > 0$. Denote this point by $r(x)$.



We thus get a map $r\colon \mathbb{D}^2 \to S^1$ such that $r(x) = x$ for $x \in S^1$. We next show that this map is continuous, and thus gives a retraction. Indeed, the function r is given by $r(x) = L_x(t_*)$, where t_* is the positive solution to the quadratic equation

$$|(1-t)f(x) + tx|^2 = 1.$$

Writing this out, we get a quadratic equation with precisely two solutions, only one of which is positive. From the explicit expression for the solution of a quadratic equation, it follows that such a t_* depends continuously on the coefficients of the equation, which in turn depend continuously on x and $f(x)$. It follows that $r\colon \mathbb{D}^2 \to S^1$ is continuous. $\square$

Exercise 13.3. Show that Theorem 13.2 still holds if we replace $(\mathbb{D}^2, S^1)$ with a pair of spaces (X, A) such that $A \subset X$, $X \cong \mathbb{D}^2$, and $A \cong S^1$. Hence, Theorem 13.2 also holds for maps $f \times I^2 \to I^2$ (where $I^2 = I \times I$ is the square).

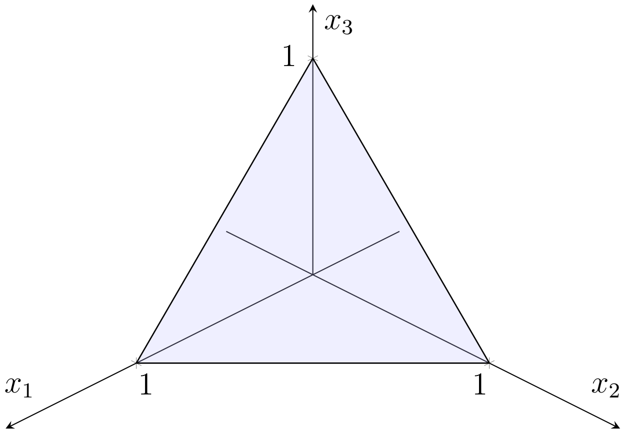
13.2 Applications of the Brouwer Fixed Point Theorem

An application of the Brouwer Fixed Point Theorem is to eigenvectors. Recall that an eigenvector of a matrix A is a vector v such that $Av = \lambda v$; the λ is called the eigenvalue. The following is a special case of the famous **Perron-Frobenius** Theorem, which has many applications, for example in graph theory or the theory of Markov chains.

Proposition 13.4. *Let $A \in \mathbb{R}^{3\times 3}$ be a matrix with only positive entries. Then A has an eigenvalue v consisting of only positive entries.*

Proof. Consider the **simplex**

$$\Delta = \{x \in \mathbb{R}^3 \colon x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 = 1\}.$$



Note that for any non-zero $x \in \mathbb{R}^3$ with non-negative entries, $x/\|x\|_1 \in \Delta$, where $\|x\|_1 = |x_1| + |x_2| + |x_3|$ is the 1-norm. Define the map

$$\varphi\colon \Delta \to \Delta, \quad x \mapsto \frac{Ax}{\|Ax\|_1}.$$

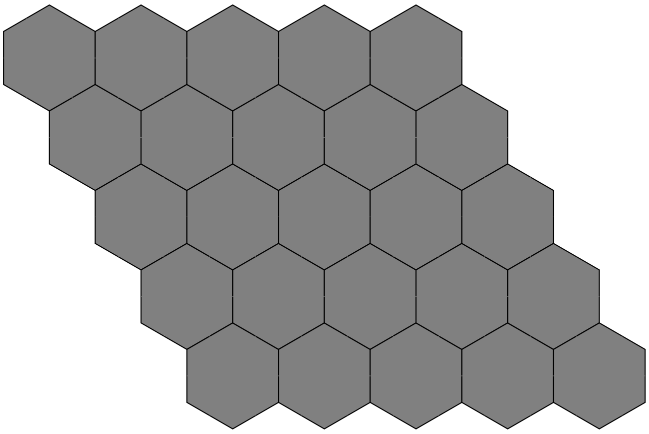
Since all the entries of A are positive, and all the entries of x are non-negative, all the coordinates of Ax are also positive, and we get a well-defined map from Δ to itself. Since $\Delta \cong \mathbb{D}^2$ (exercise: show this!), this map has a fixed point: there exists $v \in \Delta$ such that $\varphi(v) = v$. For such a v, we have

$$Av = \|Av\|_1 v,$$

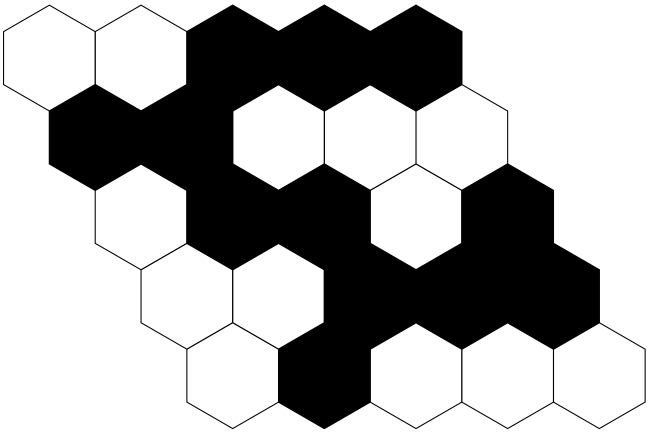
from which it follows that v is an eigenvalue. Since v is in the image of φ, its entries are all positive. □

Disclaimer: The following treatment of the game HEX is strictly optional, this is not exam material.

Another application is to the game of HEX, invented by Piet Hein and John Nash independently in the 1940. The playing board consists of an $n \times n$ grid of hexagonal tiles.



Two players alternate in colouring hexagonal tiles black or white. The goal of the first player is to find a connected path from the bottom to the top consisting of black tiles, while the goal of the second player is to find a path from the left to the right consisting of white tiles. Assuming the players manage to fill the board, the result will look something like this:



It should be intuitively clear (and easy to show) that if there is a path of one colour connecting the bottom to the top end, then there cannot be a path of the opposite color connecting the left to the right end, and vice versa. What is less clear is that there is no possibility of having a draw:

on of the players always wins. Put differently, it is impossible to colour the tiles black and white in such a way that no path exists connecting opposing sides.

Theorem 13.5. *Given an* $n \times n$ *board of hex with all the tiles coloured either black or white, there exists a path of one colour connecting the either the top and bottom sides, or the left and right sides.*

A proof of this Theorem using the Brouwer Fixed Point Theorem (and, in fact, a proof of the Fixed Point Theorem using this), can be found in the paper "Using Brouwer's fixed point theorem" by Björner, Matoušek and Ziegler.

Lecture 14

Another important application is the Borsuk-Ulam Theorem, which often goes hand-in-hand with the Brouwer Fixed Point Theorem.

14.1 The Borsuk-Ulam Theorem

Theorem 14.1. *(Borsuk-Ulam) Let* $f\colon S^2 \to \mathbb{R}^2$ *be a map. Then there exists a point* $x \in S^2$ *with* $f(x) = f(-x)$.

The theorem thus states that for any continuous function from the sphere to $\mathbb{R}^2$ there are two antipodal points for which the function has the same value. One can interpret this as saying, for example, that there are always two antipodal points on the earth's surface with equal temperature and equal pressure (assuming these two are continuous functions). The theorem, which also holds in dimension $n \geq 2$, was first proven by Karol Borsuk, who in turn attributes the problem formulation to Stanislaw Ulam. It has remarkable ramifications and applications, an overview of which can be found in the book "Using the Borsuk-Ulam Theorem" by Jiři Matoušek.

To prove the Borsuk-Ulam Theorem we need a series of auxiliary results, which are interesting in their own right. These relate to the concepts of even and odd maps, and null homotopy.

An **involution** is a map $h\colon X \to X$ such that $h(h(x)) = x$. In this case, we often write $h(x) = -x$. Typical examples of spaces with involution are the spaces $\mathbb{D}^n$, S^n or $\mathbb{R}^n$, with $-x$ just the additive inverse of x.

Definition 14.2. Let X, Y be spaces with involution. A map $f\colon X \to Y$ is called **odd** if $f(-x) = -f(x)$, and **even** if $f(-x) = x$ for all $x \in X$.

Clearly, a map does not need to be either odd or even.

Example 14.3. The map $p_2\colon S^1 \to S^1$, $z \mapsto z^2$ is even. The sine function is odd, while the cosine function is even. The identity map $\mathrm{Id}_{\mathbb{R}^n}$ is odd.

Exercise 14.4. Show that the composition of odd maps is odd and that the composition of even maps with either even or odd maps is even.

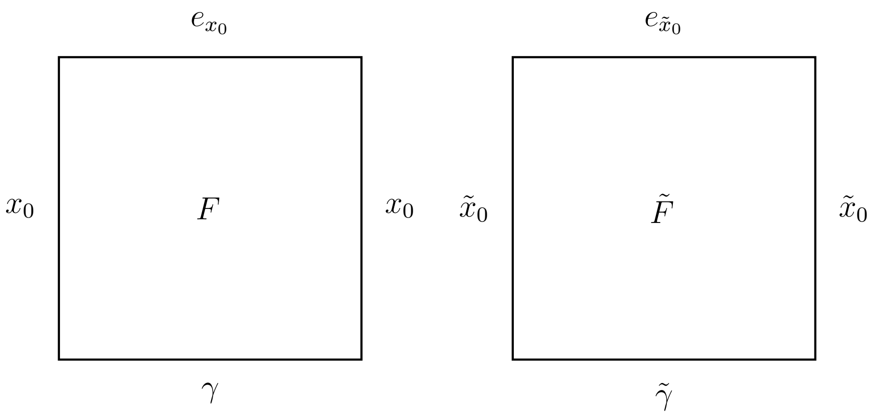
Definition 14.5. A map $f\colon X \to Y$ is called **null-homotopic** if f is homotopic to a constant map. A pointed map $f\colon (X, x_0) \to (Y, y_0)$ is called null-homotopic relative to the basepoint if there is a homotopy $f\colon X \times I \to Y$ such that $f_0 = f$ and $f_1 = e_{y_0}$, with $f_t(x_0) = y_0$.

If f is null-homotopic relative to the basepoint, we write $f \overset{x_0}{\simeq} e$. If a map $f\colon X \to Y$ is homotopic to a constant map e_{y_0}, and if x_0 is such that $f(x_0) = y_0$, then this does not necessarily mean that the pointed map $f\colon (X, x_0) \to (Y, y_0)$ is null-homotopic. The added requirement is that each map f_t in the homotopy should map x_0 to y_0.

We will first show that odd maps from S^1 to S^1 cannot be null-homotopic. We will then use this to show that any odd map from S^2 to $\mathbb{R}^2$ has to have a root, and finally use this to establish Borsuk-Ulam Theorem, by noting that for a function $f\colon S^2 \to \mathbb{R}^2$, the function $f(x) - f(-x)$ is odd. Before we begin, we state a lemma that will be useful on several occasions.

Lemma 14.6. *Let $p\colon \tilde{X} \to X$ be a covering and let $\gamma\colon I \to X$ be a loop such that $\gamma \overset{\partial}{\simeq} e_{x_0}$. Let $\tilde{x}_0 \in p^{-1}(x_0)$ and $\tilde{\gamma}$ the lift of γ with $\tilde{\gamma}(0) = \tilde{x}_0$. Then $\tilde{\gamma} \overset{\partial}{\simeq} e_{\tilde{x}_0}$.*

Proof. Let $F\colon I \times I \to X$ be a homotopy with $f_0 = \gamma$ and $f_1 = e_{x_0}$. By the homotopy lifting property, there is a unique homotopy $\tilde{F}\colon I \times I \to \tilde{X}$ with $\tilde{f}_0 = \tilde{\gamma}$.



The paths $F(0,t) = x_0$ (left boundary), $F(1,t) = x_0$ (right boundary) and $F(s,1) = x_0$ (upper boundary) are all constant paths. By Problem (4.7) (or the path lifting property), constant paths lift to constant paths, which implies that $\tilde{\gamma}$ is a loop at $\tilde{x}_0$ that is is homotopic to the constant loop $e_{\tilde{x}_0}$ via $\tilde{F}$. ☐

Proposition 14.7. *If $f\colon S^1 \to S^1$ is odd, then f is* not *null-homotopic.*

Proof. Assume that f is odd and that f is null-homotopic to a constant map, which without lack of generality we can assume to be e_1 (exercise: why?), via a homotopy $F\colon S^1 \times I \to S^1$. Consider the cover $p\colon \mathbb{R} \to S^1$, $s \mapsto \exp(2\pi i s)$. We proceed in two steps.

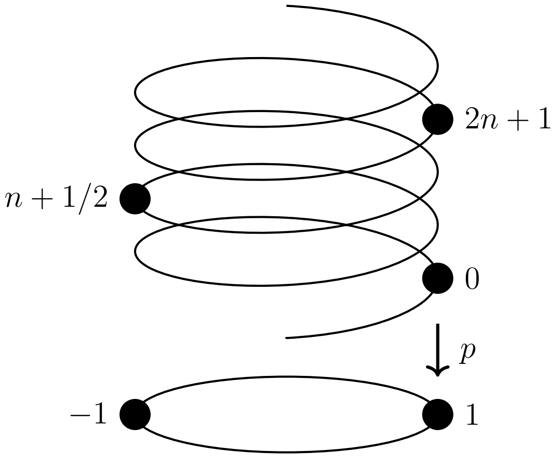
(1) Use the oddity of f to construct a loop $\gamma\colon I \to S^1$ based at 1 in such a way that γ lifts to a path $\tilde{\gamma}$ from 0 to an odd endpoint $2n+1$.

Set $g = f/f(1)$. Clearly, this is again odd and has the property that $g(1) = 1$. We can thus define a loop $\gamma\colon I \to S^1$ by setting $\gamma(s) = g(e^{2\pi i s})$. Since

$$e^{i\pi} = -1,$$

we get that $\gamma(s + 1/2) = g(\exp(2\pi i s + \pi i)) = -\gamma(s)$ for $s \in [0, 1/2]$, where we used the fact that g is odd. In particular, $\gamma(1/2) = -1$. By applying path-lifting to γ, we get a curve

$\tilde{\gamma}\colon I \to \mathbb{R}$ with $\tilde{\gamma}(0) = 0$ and $\tilde{\gamma}(1/2) = n + 1/2$ for some $n \in \mathbb{Z}$, since $\gamma(1/2) = -1$ and $p^{-1}(-1) = \{m + 1/2\colon m \in \mathbb{Z}\}$. We would like to show that, as we wind on up that road, we arrive at $\tilde{\gamma}(1) = 2n + 1$.



Consider the two paths $\alpha, \beta\colon I \to \mathbb{R}$ given by $\alpha(s) = n + 1/2 + \tilde{\gamma}(s/2)$ and $\beta(s) = \tilde{\gamma}((s+1)/2)$. We then have

- $p \circ \alpha(s) = -\gamma(s/2) = \gamma((s+1)/2) = p \circ \beta$;
- $\alpha(0) = \beta(0) = n + 1/2$.

By the uniqueness of lifts, it follows that $\alpha = \beta$, and hence

$$2n + 1 = \alpha(1) = \beta(1) = \tilde{\gamma}(1).$$

This establishes the first part.

(2) Show that the lifted path $\tilde{\gamma}$ is a loop that is homotopic to the constant loop at 0.

Using the fact that F is null-homotopic, we construct a homotopy from γ to e_{x_0} as follows:

$$\begin{aligned}\Gamma\colon I \times I &\to S^1 \\ (s,t) &\mapsto \frac{F(e^{2\pi i s}, t)}{F(1,t)},\end{aligned}$$

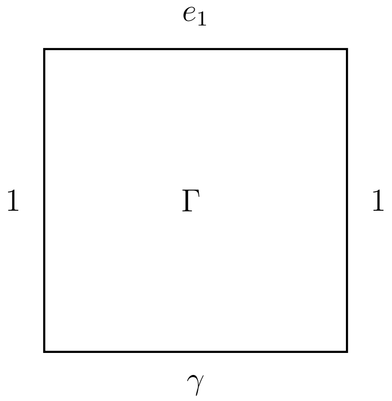
where we use the division over the complex numbers. Clearly, at the boundaries

$$\Gamma(s,0) = \gamma(s), \quad \Gamma(s,1) = 1, \quad \Gamma(0,t) = \Gamma(1,t) = 1.$$

Thus Γ is a homotopy from γ to the constant loop at 1, and by Lemma 14.6, $\tilde{\gamma}$ is a null-homotopic loop. Therefore,

$$0 = \tilde{\gamma}(0) = \tilde{\gamma}(1) = 2n + 1,$$

which is not possible if $n \in \mathbb{Z}$. We get a contradiction to the assumption that f is null-homotopic, completing the proof. $\square$



Corollary 14.8. *If $f\colon S^2 \to \mathbb{R}^2$ is odd, then there exists $x \in S^2$ such that $f(x) = 0$.*

Proof. Assume that f is odd, and that $f(x) \neq 0$ for all $x \in S^2$. The idea is to use f to define maps $g\colon S^1 \times I \to S^2$ and $p\colon S^2 \to S^1$, such that the composition $H = p \circ g\colon S^1 \times I \to S^1$ is a homotopy from an odd to a constant map, in contradiction to Proposition 14.7.

$$\begin{tikzcd}
S^1 \times I \arrow[rr, "H"] \arrow[dr, "g"] & & S^1 \\
& S^2 \arrow[ur, "p"] &
\end{tikzcd}$$

Define the map

$$\begin{aligned} g\colon S^2 &\to S^1, \\ x &\mapsto \frac{f(x)}{\|f(x)\|}. \end{aligned}$$

Since $g(-x) = f(-x)/\|f(-x)\| = -f(x)/\|f(x)\|$, g is again odd. Define the upper hemisphere

$$U = \{(x, y, z) \in S^2 \colon z \geq 0\},$$

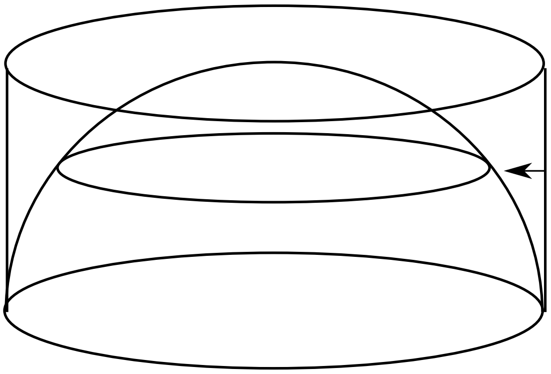
and a map $p\colon S^1 \times I \to U$ by setting

$$p(e^{i\theta}, t) = (t\cos(\theta), t\sin(\theta), \sqrt{1 - t^2}).$$

It follows that p_0 is constant and p_1 embeds the circle S^1 into the equator $E = S^1 \times \{0\} \subset S^2$ (note that here, as usual, we identify $\mathbb{C}$ with $\mathbb{R}^2$ and $e^{i\theta}$ with $(\cos(\theta), \sin(\theta))$, and freely alternate between these representations). Finally, consider the homotopy

$$H\colon S^1 \times I \to S^1, \quad H(e^{i\theta}, t) = g \circ p.$$

Then $h_0 = g \circ p_0$ is a constant map and $h_1 = g \circ p_1$ is odd: this follows from the fact that both g and p_1 are odd. We therefore have a homotopy from an odd map to a constant map, in contradiction to Proposition 14.7. $\square$



Proof of Theorem 14.1. Define the map

$$g(x) = f(x) - f(-x).$$

By definition, this is an odd map, so by Corollary 14.8 this has a zero, i.e., there exists an x such that $f(x) - f(-x) = 0$. $\square$

Lecture 15

In this lecture we will compute the fundamental group of the torus (in any dimension) and of the n-dimensional sphere, for $n \geq 2$. As a consequence, we will see that

$$\mathbb{T}^n \not\cong S^n.$$

15.1 Product spaces

The proof of the following is left as an exercise. The main ingredient is the observation that a map $f \colon Z \to X \times Y$ is continuous if and only if the compositions $p_X \circ f$ and $p_Y \circ f$ are continuous, where $p_X \colon X \times Y \to X$ and $p_Y \colon X \times Y \to Y$ are the projections onto X and Y, respectively.

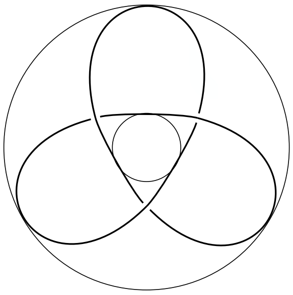
Proposition 15.1. *Let (X, x_0) and (Y, y_0) be pointed spaces. Then*

$$\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Example 15.2. Consider the torus $\mathbb{T}^2 = S^1 \times S^1$. Then

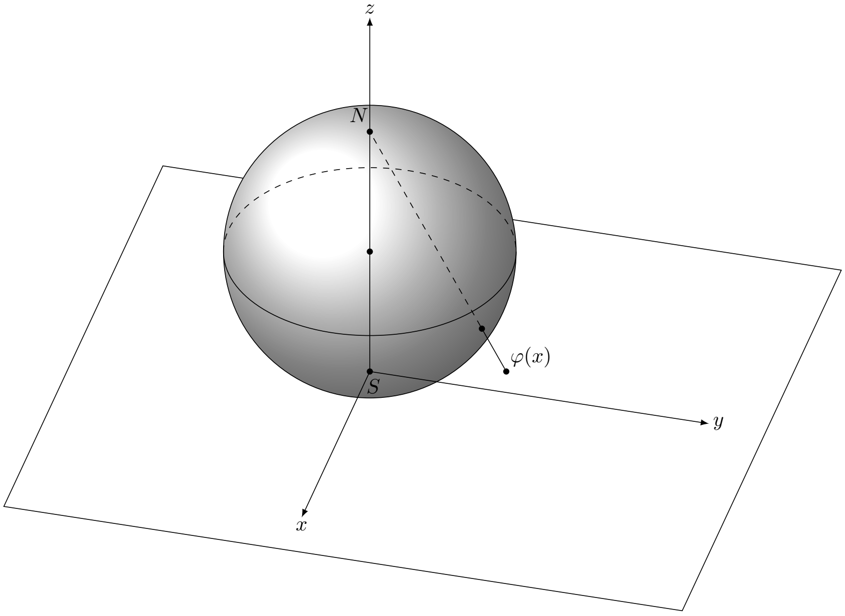
$$\pi_1(\mathbb{T}^2, (1,1)) = \mathbb{Z} \times \mathbb{Z}.$$

There are two types of loops in the embedded torus: that the fundamental group of the torus is $\mathbb{Z}^2$ should therefore not be surprising. Consider for example a loop that winds around one circle of the torus three times, and two times around the other. The resulting path is the trefoil know, one of many torus knots.



15.2 The fundamental group of the sphere S^n

The computation of the fundamental group of S^n for $n \geq 2$ uses the stereographic projection. Fix a point on S^n, for example the "north pole" $N = (0, \ldots, 0, 1)$.



The stereographic projection is a continuous map $\varphi\colon S^n - \{N\} \to \mathbb{R}^n$, by mapping a point x to the intersection of the line joining N and x with the hyperplane perpendicular to N touching the south pole $S = -N$.

Exercise 15.3. Derive the precise form of φ and its inverse φ^{-1}, and show that these are continuous maps.

Proposition 15.4. *For $n \geq 2$ and $x_0 \in S^n$, $\pi_1(S^n, x_0) \cong \{0\}$.*

Proof. As the stereographic projection gives a homeomorphism from the open set $U_1 = S^n - \{N\}$ to $\mathbb{R}^n$, and similarly from $U_2 = S^n - \{S\}$ to $\mathbb{R}^n$, we see that the sphere can be written as a union of open sets

$$S^n = U_1 \cup U_2,$$

with $U_1 \cong \mathbb{R}^n$ and $U_2 \cong \mathbb{R}^n$. In addition, also using the stereographic projection, we see that $U_1 \cap U_2 \cong \mathbb{R}^n - \{0\}$, which is path-connected. Assume without lack of generality that $x_0 \in U_1 \cap U_2$ (as S^n is path-connected, the fundamental groups with different basepoints are all isomorphic).

Given a loop $\gamma\colon I \to S^n$, we have a cover $I = \gamma^{-1}(U_1) \cup \gamma^{-1}(U_2)$. By the Lebesgue covering lemma, we can find a subdivision $0 = t_0 < t_1 < \cdots < t_m = 1$ such that for every subinterval we have $\gamma([t_{i-1}, t_i]) \subset U_1$ or $\gamma([t_{i-1}, t_i]) \subset U_2$. Set $\gamma_i := \gamma|_{[t_{i-1}, t_i]}$ for $1 \leq i \leq m$. Then

$$\gamma = \gamma_1 * \cdots * \gamma_m.$$

If $\gamma([t_{i-1}, t_i]) \subset U_j$ and $\gamma([t_i, t_{i+1}]) \subset U_k$ for $j, k \in \{1, 2\}$ (the possible cases are that $j = k$ or $j \neq k$), then there exists a path α_i in $U_j \cap U_k$ (which may just be U_1 or U_2 if the indices are equal) connecting $\gamma(t_i)$ to x_0 (since that space is path-connected). Consider now the new path

$$\beta = (\gamma_1 * \alpha_1) * (\overline{\alpha}_1 * \gamma_2 * \alpha_2) * \cdots * (\overline{\alpha}_{m-1} * \gamma_m).$$

Since each of the $\alpha_{i-1} * \gamma_i * \overline{\alpha}_i$ is a loop, β is a concatenation of loops. Moreover, each of these loops is contained in one of U_1 or U_2 (or both), and since these spaces are homeomorphic to $\mathbb{R}^n$, each of these loops is homotopic to the constant loop e_{x_0}. Therefore,

$$\beta \overset{\partial}{\simeq} e_{x_0}.$$

But we also have that

$$\gamma \overset{\partial}{\simeq} \beta,$$

using a homotopy that moves each $\alpha_i * \overline{\alpha}_i$ to e_{x_0}. It follows that $[\gamma] = [e_0]$, and hence the fundamental group is the trivial group. □

Exercise 15.5. Find out where the argument breaks down for $n = 1$.

Corollary 15.6. *The n-torus $\mathbb{T}^n$ is not homotopic to the sphere S^n.*

Lecture 16

In this lecture we look at the relationship between isomorphism classes of covers and subgroups of the fundamental group. This is what is also known as **Galois correspondence**, due to its analogy to Galois theory, where one has field extensions instead of coverings and the Galois group instead of the fundamental group.

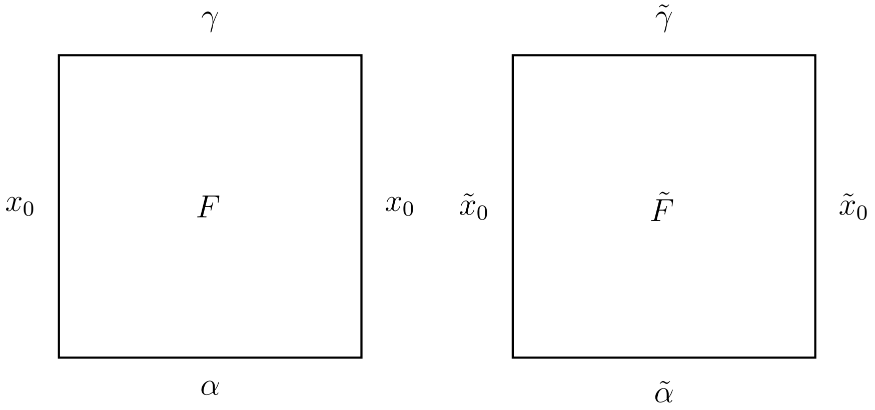
Proposition 16.1. *Let* $p\colon \tilde{X} \to X$ *be a covering,* $x_0 \in X$*, and* $\tilde{x}_0 \in p^{-1}(x_0)$*. Then:*

(a) The induced homomorphism $p_*\colon \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ *is injective;*

(b) If $[\alpha] \in \pi_1(X, x_0)$ *and* $\tilde{\alpha}$ *is the lift of* α *with* $\tilde{\alpha} = \tilde{x}_0$*, then* $\tilde{\alpha}$ *is a loop (i.e.,* $\tilde{\alpha}(1) = \tilde{x}_0$*) if and only if* $[\alpha] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$*.*

Proof. **(a)** Assume that $[\tilde{\alpha}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ maps to $[e_{x_0}]$, i.e., that $p \circ \tilde{\alpha}(0) \overset{\partial}{\simeq} e_{x_0}$. Then by Lemma 14.6 (Lecture 14) we have that $\tilde{\alpha} \overset{\partial}{\simeq} e_{\tilde{x}_0}$.

(b) Clearly, if $\tilde{\alpha}$ is a loop, then $[\alpha] = p_*([\tilde{\alpha}])$, which shows the "only if" direction. For the "if" direction, assume that $[\alpha] = p_*([\tilde{\gamma}])$ for some $[\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_0)$. This means $\alpha = p \circ \tilde{\alpha} \overset{\partial}{\simeq} p \circ \tilde{\gamma} = \gamma$. Just as in the proof of Proposition 14.6, we get a homotopy from α to γ that fixes endpoints, and that lifts to a homotopy from $\tilde{\alpha}$ to $\tilde{\gamma}$.



As the left and right boundaries, $F(0,t)$ and $F(1,t)$, are constant (x_0), these lift to constant paths. Since the upper boundary $F(s,1) = \tilde{\gamma}(s)$ is a loop at $\tilde{x}_0$, this means that the whole left and right boundaries are $\tilde{x}_0$, and therefore that $\tilde{\alpha}$ is a loop. ☐

The proposition shows that the image $p_*(\pi_1(\tilde{X}, \tilde{x}_0)$ is a subgroup of $\pi_1(X, x_0)$ that is isomorphic to $\pi_1(\tilde{X}, \tilde{x}_0)$. This may seem counterintuitive at first, as covering maps are (generally) surjective.

Example 16.2. Consider the d-fold covering $p_d\colon S^1 \to S^1$. Identifying the fundamental groups with $\mathbb{Z}$, the induced map is $n \mapsto d \cdot n$. Hence, $(p_d)_*(\pi_1(S^1, 1) \cong d\mathbb{Z}$.

Recall that if G is a group and $H \leq G$ a subgroup, the **index** of H in G, $[G\colon H]$, is the number of right-cosets $G/H = \{Hg\}_{g \in G}$.

Definition 16.3. Let $p\colon \tilde{X} \to X$ be a covering and assume that $\tilde{X}$ and X are path-connected. Then for any $x \in X$,

$$\deg(p) := |p^{-1}(x)|$$

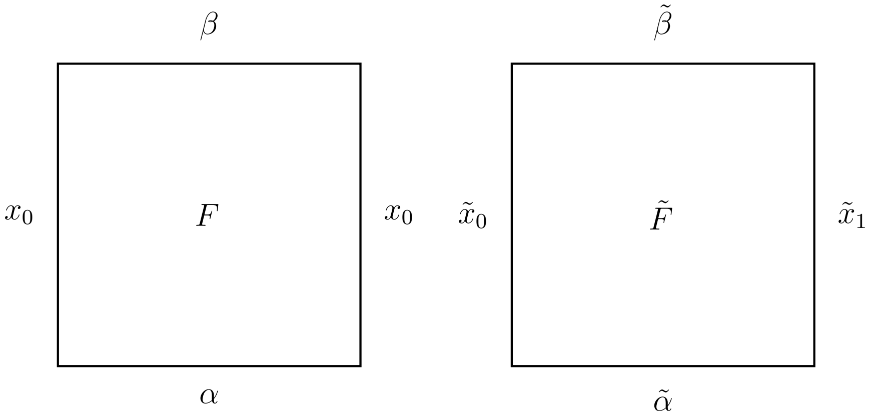
is called the **degree** of the covering.

Exercise 16.4. Show that this is well-defined. Verify whether the conditions of path-connectedness and the connectedness of $\tilde{X}$ can be relaxed.

Proposition 16.5. *Let $p\colon \tilde{X} \to X$ be a covering and assume that $\tilde{X}$ and X are path-connected. Let $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$. Then*

$$\deg(p) = [\pi_1(X, x_0)\colon p_*(\pi_1(\tilde{X}, \tilde{x}_0))].$$

Proof. Set $G = \pi_1(X, x_0)$ and $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Let $[\alpha] \in G$ and $\tilde{\alpha}$ a lift of α, starting at $\tilde{x}_0$ and ending at $\tilde{x}_1 \in p^{-1}(x_0)$. If β is another loop with $[\alpha] = [\beta]$, then by the same argument as in the proof of Proposition 16.1(b) (see the figure), β lifts to a path $\tilde{\beta}$ with the same endpoint $\tilde{x}_1$, so that the endpoint only depends on the class of the loop, and not the representative.



Let $[h] \in H$. Then by Proposition 16.1(b), h lifts to a loop $\tilde{h}$ in $\tilde{X}$, and $h * \alpha$ lifts to a path $\tilde{h} * \tilde{\alpha}$ from $\tilde{x}_0$ to $\tilde{x}_1$. From this we get a map from the set of right-cosets $H[\alpha]$ to the endpoints of lifts $\tilde{\alpha}$:

$$\begin{aligned} \Phi\colon G/H &\to p^{-1}(x_0) \\ H[\alpha] &\mapsto \tilde{\alpha}(1). \end{aligned}$$

We need to show that this map is injective and surjective.

For the injectivity, assume that $\Phi(H[\alpha]) = \Phi(H[\beta])$. Then $\tilde{\alpha}(1) = \tilde{\beta}(1)$, and hence $\tilde{\alpha} * \overline{\tilde{\beta}}$ is defined and is a loop at $\tilde{x}_0$ in $\tilde{X}$. It follows that

$$p_*([\tilde{\alpha} * \overline{\tilde{\beta}}]) = [p \circ \tilde{\alpha} * \overline{\tilde{\beta}}] = [p \circ \tilde{\alpha} * p \circ \overline{\tilde{\beta}}] = [\alpha] \bullet [\overline{\beta}] \in H,$$

from which we get that

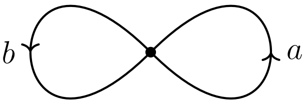
$$H[\alpha] = H[\alpha][\overline{\beta}][\beta] = H[\beta].$$

For the surjectivity, we see that since $\tilde{X}$ is path connected, there exists a path from $\tilde{x}_0$ to any other point in $p^{-1}(x_0)$. Each such path projects to a loop α in X, and Φ maps the corresponding element $H[\alpha]$ in G/H to $\tilde{\alpha}(1)$. Therefore, Φ is a bijection. $\square$

Lecture 17

In this lecture we will study free products of groups, a construction that is important in the study of the fundamental group of various spaces.

Consider for example the figure-eight

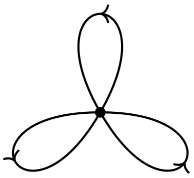


Definition 17.1. Let $\{(X_\alpha, x_\alpha)\}_\alpha$ be a collection of pointed topological spaces. The **wedge sum** of this collection is defined as

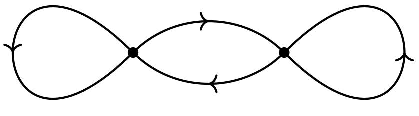
$$\bigvee_\alpha (X_\alpha, x_\alpha) = \bigsqcup_\alpha X_\alpha / (x_\alpha \sim x_\beta),$$

that is, the disjoint union of the X_α with the points x_α all identified.

Example 17.2. The figure-eight is $S^1 \vee S^1$ (we omit the basepoints from the notation when it is not important). The set bouquet is given by $S^1 \vee S^1 \vee S^1$.



$S^1 \vee S^1 \vee S^1$



$(S^1 \vee S^1) \vee S^1$

Note that $S^1 \vee S^1 \vee S^1$ is not the same as $(S^1 \vee S^1) \vee S^1$! The basepoints do play a role, in the latter we are identifying only two of them.

If we denote the two loops in the figure-eight $S^1 \vee S^1$ by a and b, and their inverses at $\overline{a}$ and $\overline{b}$, then intuitively, every loop can be written as a "word", for example

$$aaab\overline{b}b\overline{a}a\overline{a}bb$$

This means: go around a 3 times, then around b, around b backwards and again around b, etc. One can also see that a loop described like this can be "reduced" to a homotopic loop: loops of the form $b\overline{b}$ are homotopic to the constant loop, so we can replace them with e, and we can then remove the constant loops from the expression. It follows that the loop above is homotopic to a reduced loop of the form

$$a^3b\overline{a}b^2 \text{ or } a^3ba^{-1}b^2$$

From this observation one can conjecture the form of the fundamental group of the figure-eight: take two circles $A = S^1$ and $B = S^1$, joint them at a point x_0, take generators a and b of the fundamental groups of A and B (which are isomorphic to $\mathbb{Z}$), and describe the fundamental group of the figure-eight as the set of reduced words on a and b, where the inverse of a word is the word obtained by changing the order of letters and replacing every letter with its inverse (for example, the inverse of aba^2b^{-3} is $b^3a^{-2}b^{-1}a^{-1}$), and taking as multiplication the concatenation of words, followed by a reduction. We formalize this process next, using the concept of free product.

17.1 The free product of groups

Definition 17.3. Let $\{G_\alpha\}_\alpha$ be a collection of groups. A **word** on these groups is a finite sequence $g_1 \cdots g_m$ of elements of the $g_i \in G_{\alpha_i}$, and m is the length of the word. The empty word is denoted by ϵ. The **product** of two words is the concatenation,

$$(g_1 \cdots g_m) * (h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n.$$

Definition 17.4. A word $g = g_1 \cdots g_m$ is called **reduced** if $g_i \neq e_{\alpha_i}$ (the unit element of the group G_{α_i}) and for any two consecutive letters g_i, g_{i+1}, $\alpha_i \neq \alpha_{i+1}$ (that is, consecutive letters are not from the same group).

Given any word g on the groups $\{G_\alpha\}_\alpha$, we can **reduce** it to a reduced word g' as follows.

(a) If $g_i = e_{\alpha_i}$, then remove it from g;

(b) If $\alpha_i = \alpha_{i+1}$, then replace $g_i g_{i+1}$ with the group element $g_i \cdot g_{i+1}$ from G_{α_i}.

As every such operation reduces the length of the word by one, the process has to terminate. Moreover, a word is reduced if and only it can't be reduced further by the above two operations.

Remark 17.5. A word g can be reduced to a word g' in different ways, depending on the order in which the operations are applied. It is not yet obvious that every word reduces to a *unique* reduced word.

On the set of reduced words we can define a multiplication as follows. Given reduced words $g = g_1 \dots g_m$ and $h = h_1 \cdots h_n$, construct a new reduced word $g \bullet h$ by taking the concatenation $g * h$, and then reducing the word recursively as follows:

$$g \bullet h = \begin{cases} g * h & \text{if } g_m, h_1 \text{ not in same group,} \\ g_1 \cdots g_{m-1}(g_m \cdot h_1)h_2 \cdots h_n & \text{if } g_m, h_1 \in G_\alpha \text{ and } g_m \cdot h_1 \neq e_\alpha \\ g_1 \cdots g_{m-1} \bullet h_2 \cdots h_n & \text{if } g_m \cdot h_1 = e_\alpha. \end{cases}$$

where $*$ is the concatenation of words.

This process eventually leads to a reduced word, denoted by $g \bullet h$. Define the set

$$*_\alpha G_\alpha = \{ \text{ reduced words on } \{G_\alpha\}_\alpha \}$$

Theorem 17.6. *The pair $(*_\alpha G_\alpha, \bullet)$ is a group, called the* ***free product*** *of $\{G_\alpha\}_\alpha$. The unit element is the empy word $\epsilon = []$, and the inverse of an element $g_1 \cdots g_m$ is $g_m^{-1} \cdots g_1^{-1}$.*

Checking that the inverse has the given form is straight-forward. Checking associativity of the operation requires some work.

Proof. The verification that $g \bullet h$ is again a reduced word follows from the definition: if the concatenation $g * h$ is not reduced, then it is replaced by a shorter word. As words have finite length, this process has to terminate in a reduced word. That the empty word is the unit element follows from the definition of the product $\bullet$. That the inverse element has the given form is obvious, but can be shown formally by induction: if $m = 1$, then $g_1 \bullet g_1^{-1} = \epsilon$ (by the definition of the product) and assuming the statement holds for $m - 1$, then

$$g_1 \cdots g_{m-1} g_m \bullet g_m^{-1} g_{m-1}^{-1} \cdots g_1^{-1} = g_1 \cdots g_{m-1} g_{m-1}^{-1} \cdots g_1^{-1} = \epsilon.$$

To have a group structure, what remains is to show associativity, namely that for reduced words g, h, k we have

$$(g \bullet h) \bullet k = g \bullet (h \bullet k).$$

To prove this, set $W = *_\alpha G_\alpha$ for the set of reduced words, and consider the group of bijections $\mathrm{Sym}(W)$. We will "embed" W into $\mathrm{Sym}(W)$ via an injective map L that is compatible with multiplication, i.e., $L(g \bullet h) = L(g) \circ L(h)$, and from this the associativity in $\mathrm{Sym}(W)$ will naturally lead to the associativity of the product $\bullet$ in W.

To start with, for every element $g \in G_\alpha$ we have a map L_g, the *left multiplication*, such that $L_g(h) = g \bullet h$ for a word $h \in W$. If $g_1, g_2 \in G_\alpha$ and $h = h_1 \dots h_m$, then one easily verifies that

$$L_{g_1} \circ L_{g_2} = L_{g_1 g_2}, \tag{17.1}$$

and hence that $L_{g^{-1}} = L_g^{-1}$, so that $L_g \in \mathrm{Sym}(W)$. For any word $g = g_1 \cdots g_m$, the map

$$\begin{aligned} L \colon W &\mapsto \mathrm{Sym}(W) \\ g &\mapsto L_{g_1} \circ \cdots \circ L_{g_m} =: L_{g_1 \cdots g_m} \end{aligned}$$

is injective, since for any $g \in W$, $L_g(\epsilon) = g$, hence if $g \neq h$ in W, $L_g \neq L_h$ in $\mathrm{Sym}(W)$.

Note that, by (17.1), the composition $L_g \circ L_h$ obeys the same rules as the product $\bullet$: if $g = g_1 \cdots g_m$ and $h = h_1 \cdots h_n$ are reduced words, then

$$L_g \circ L_h = \begin{cases} L_{g*h} & \text{if } g_1, h_m \text{ not in same group,} \\ L_{g_1} \circ \cdots \circ L_{g_{m-1}} \circ L_{g_m h_1} \circ L_2 \circ \cdots \circ L_n & \text{if } g_m, h_1 \in G_\alpha \text{ and } g_m h_1 \neq e_\alpha \\ L_{g_1} \circ \cdots \circ L_{g_{m-1}} \circ L_{h_2} \circ \cdots \circ L_n & \text{if } g_m \cdot h_1 = e_\alpha. \end{cases}$$

From this it follows that $L_{g \bullet h} = L_g \circ L_h$.

The associativity now follows from

$$\begin{aligned} L_{(g \bullet h) \bullet k} &= L_{g \bullet h} \circ L_k \\ &= (L_g \circ L_h) \circ L_k \\ &= L_g \circ (L_h \circ L_k) \\ &= L_g \circ L_{h \bullet k} \\ &= L_{g \bullet (h \bullet k)}. \end{aligned}$$

By the injectivity of L, $(g \bullet h) \bullet k = g \bullet (h \bullet k)$. ☐

A consequence of the associativity is that the order of reduction does not affect the end result: every word reduces to a unique reduced form.

Example 17.7. Consider two copies of the group $\mathbb{Z}_2 = \mathbb{Z}/(2\mathbb{Z})$, with generators a and b, respectively. Since $a^2 = e$ and $b^2 = e$, all the reduced words consist of alternating sequences of a and b, for example $ababab$ or $babab$. The inverse of ab is ba, and therefore the set of words of even length forms a cyclic subgroup $G \cong \mathbb{Z}$ generated by ab. If $H \cong \mathbb{Z}_2$ is the subgroup generated by a, then $\mathbb{Z}_2 * \mathbb{Z}_2 = GH$; that is, $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$, the semi-direct product of these two subgroups.

Lecture 18

Note that every group G_α is a subgroup of $*_\alpha G_\alpha$, via the inclusion that maps g to the word consisting only of g for $g \neq e$, and e to the empty word. Let

$$\iota_\alpha \colon G_\alpha \hookrightarrow *_\alpha G_\alpha$$

denote this inclusion. The free product of a collection of groups $\{G_\alpha\}_\alpha$ satisfies the following universal property.

Lemma 18.1. *Let $\{\varphi_\alpha\}_\alpha$ be a collection of group homomorphisms $\varphi_\alpha \colon G_\alpha \to G$. Then there exists a unique map*

$$*_\alpha\varphi_\alpha \colon *_\alpha G_\alpha \to G$$

such that $(*_\alpha\varphi_\alpha) \circ \iota_\alpha = \varphi_\alpha$.

Proof. Define

$$(*_\alpha\varphi_\alpha)(g_1 \cdots g_m) = \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_m}(g_m), \tag{18.1}$$

where we assumed that $g_i \in G_{\alpha_i}$. This clearly satisfies the property $*_\alpha\varphi_\alpha \circ \iota_\alpha = \varphi_\alpha$. Moreover, since every φ_α is a group homomorphism, for $g_i, g_{i+1} \in G_\alpha$ we get $\varphi(g_i)\varphi(g_{i+1}) = \varphi(g_i g_{i+1})$ and $\varphi(e_\alpha) = e$, so that $*_\alpha\varphi_\alpha$ is compatible with the operations bringing a word into reduced form. Therefore, $*_\alpha\varphi_\alpha(g \bullet h) = \varphi(g)\varphi(h)$ and we have a group homomorphism. The requirement that the restriction to the G_α satisfies $*_\alpha\varphi_\alpha \circ \iota_\alpha = \varphi_\alpha$ leaves one with no other choice than to define the homomorphism as in 18.1. □

18.1 The Seifert-van Kampen Theorem

We now apply the free product to topology. The goal is to reduce the computation of the fundamental group of an open cover to the fundamental groups of the individual sets in the cover.

Let $X = \bigcup_\alpha A_\alpha$ be an open cover and denote by $\iota_\alpha \colon A_\alpha \hookrightarrow X$ and the inclusion maps. Assume that $x_0 \in \bigcap_\alpha A_\alpha$. The inclusion maps induce maps

$$(\iota_\alpha)_* \colon \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

of the fundamental groups with base x_0. By Lemma 18.1, these maps induce a map

$$\Phi = *_\alpha(\iota_\alpha)_* \colon *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0),$$

and these maps are compatible with the inclusion $i_\alpha\colon \pi_1(A_\alpha, x_0) \hookrightarrow *_\alpha\pi_1(A_\alpha, x_0)$, in that $*_\alpha(\iota_\alpha)_* \circ i_\alpha = (\iota_\alpha)_*$.

It is relatively easy to show that if the pairwise intersections $A_\alpha \cap A_\beta$ are path-connected, then the induced map Φ is surjective. In general, however, it will not be injective: the reason is that loops in $A_\alpha \cap A_\beta$ are accounted for twice in $*_\alpha\pi_1(A_\alpha, x_0)$, once as an element of $\pi_1(A_\alpha, x_0)$ and once as an element of $\pi_1(A_\beta, x_0)$. To remedy this, we have to factor such loops out, and for this we need to study the inclusion

$$\iota_{\alpha\beta}\colon A_\alpha \cap A_\beta \to A_\alpha,$$

with the induced maps $(\iota_{\alpha\beta})_*$ of fundamental groups. The whole setup is summarised in the following "Seifert-van Kampen" commutative diagram:

$$\begin{tikzcd}
 & \pi_1(A_\alpha, x_0) \arrow[dr] \arrow[drr, bend left, "(\iota_\alpha)_*"] & & \\
\pi_1(A_\alpha \cap A_\beta, x_0) \arrow[ur, "(\iota_{\alpha\beta})_*"] \arrow[dr, "(\iota_{\beta\alpha})_*"'] & & *_\alpha\pi_1(A_\alpha, x_0) \arrow[r, dashed, "\Phi"] & \pi_1(X, x_0) \\
 & \pi_1(A_\beta, x_0) \arrow[ur] \arrow[urr, bend right, "(\iota_\beta)_*"'] & &
\end{tikzcd} \tag{18.1}$$

Note that every $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$ is represented in $*_\alpha\pi_1(A_\alpha, x_0)$ as $(\iota_{\alpha\beta})_*(\omega)$, and as $(\iota_{\beta\alpha})_*(\omega)$. Define the set

$$U = \{(\iota_{\alpha\beta})_*(\omega)(\iota_{\beta\alpha})_*(\omega)^{-1}\colon \alpha, \beta, \omega \in \pi_1(A_\alpha \cap A_\beta, x_0)\},$$

and let N be the *normal closure* of U, i.e., the smallest normal subgroup of $*_\alpha\pi_1(A_\alpha, x_0)$ containing U. Recall that a subgroup $H \subset G$ is called *normal* if it is closed under conjugation: $gHg^{-1} = H$ for $g \in G$. We can now formulate the Seifert-van Kampen Theorem.

Theorem 18.2. *(Seifert-van Kampen) Let* $X = \bigcup_\alpha A_\alpha$ *be a cover with open sets and assume* $x_0 \in \bigcap A_\alpha$*. Then:*

(I) *If for all* α, β, $A_\alpha \cap A_\beta$ *is path-connected, then the map*

$$\Phi = *_\alpha(\iota_\alpha)_*\colon\ *_\alpha\, \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

is surjective.

(II) *If in addition for every* α, β, γ *the intersection* $A_\alpha \cap A_\beta \cap A_\gamma$ *is path-connected, then* $\ker\Phi = N$*, and hence*

$$\pi_1(X, x_0) \cong *_\alpha\pi_1(A_\alpha, x_0)/N.$$

Example 18.3. Consider the sphere S^n and the cover U_1, U_2 consisting of the open sets by removing the north and the south pole, respectively. The intersection $U_1 \cap U_2$ is path-connected, so we have a surjective map

$$\Phi\colon \pi_1(U_1, x_0) * \pi_1(U_2, x_0) \to \pi_1(S^{n-1}, x_0).$$

Since U_1 and U_2 are contractible, the fundamental groups are the trivial group, and it follows that $\pi_1(S^n, x_0)$ is also the group with one element. The argument does not extend to S^1, since $U_1 \cap U_2$ is disconnected.

Example 18.4. Let $X = \bigvee_\alpha X_\alpha = \bigsqcup_\alpha X_\alpha/(x_\alpha \sim x_\beta)$ be the wedge product of pointed topological spaces (X_α, x_α). Assume that for every α there exists an open neighbourhood U_α of x_α in X_α that is contractible, i.e., deformation retracts to x_α. For every α, define $A_\alpha = X_\alpha \vee \bigvee_{\beta\neq\alpha} U_\beta$. Every A_α is an open set in X and $x \in \bigcap_\alpha A_\alpha$, where $x = [x_\alpha]$ is the point at which the X_α are "glued together" (formally, the equivalence class containing the base points x_α). The intersection of any two distinct A_α is the set $\bigvee_\alpha U_\alpha$, which deformation retracts to x.[3] The fundamental groups of the intersections $A_\alpha \cap A_\beta$ is thus the trivial group, and by the Seifert-van Kampen Theorem, it follows that the fundamental group of X is isomorphic to the free product of the fundamental groups of the X_α:

$$\pi_1\left(\bigvee_\alpha X_\alpha, x\right) \cong *_\alpha \pi_1(X_\alpha, x_\alpha).$$

In particular, the fundamental group of the figure-eight, $S^1 \vee S^1$, is isomorphic to the free group generated by two elements a and b.

Before proving this theorem, we discuss a bit what it means. Any collection of loops $[\gamma_i] \in \pi_1(A_{\alpha_i}, x_0)$ gives rise to a loop $(\iota_{\alpha_i})_*([\gamma_i]) = [\iota_{\alpha_i}(\gamma_i)]$ in $\pi_1(X, x_0)$, and omitting the inclusion map we can simply denote it by $[\gamma_i] \in \pi_1(X, x_0)$. The induced map from the free product then looks as follows:

$$\begin{aligned}\Phi\colon\ *_\alpha\, \pi_1(A_\alpha, x_0) &\to \pi_1(X, x_0)\\ [\gamma_1]\cdots[\gamma_m] &\mapsto (\iota_{\alpha_1})_*([\gamma_1])\cdots(\iota_{\alpha_m})_*([\gamma_m]) = [\gamma_1 * \cdots * \gamma_m],\end{aligned}$$

where in the last line we consider γ_i as a loop in X. Thus for the first part, the surjectivity, we need to derive that every loop in X based at x_0 "factors" as a concatenation of loops $\gamma_1, \dots, \gamma_m$, with each of these in one A_α. This is reminiscent of the derivation of the fundamental group of S^n for $n \geq 2$.

The fact that this map is *not injective* has to do with the fact that such a factorization is not unique: if γ is a loop in $A_\alpha \cap A_\beta$, then it is represented in $\pi_1(A_\alpha, x_0)$ as $(\iota_{\alpha\beta})([\gamma])$, and in $\pi_1(A_\beta, x_0)$ as $(\iota_{\beta\alpha}([\gamma])$. Each of these can appear as a letter in a word in $*_\alpha\pi_1(A_\alpha, x_0)$, and replacing one with the other in this word will not change the image of the word under Φ: the quotient $(\iota_{\alpha\beta})([\gamma])(\iota_{\beta\alpha})([\gamma])^{-1}$ is therefore in the *kernel* of Φ. The second part of the Seifert-van Kampen Theorem thus tells us that "factoring out" this kernel gives an isomorphism.

Example 18.5. Let $X = A_\alpha \cup A_\beta$ and consider the setting of (18.1). Take $X = \mathbb{R}^2$, $A_\alpha = X - \{(1,0)\}$, $A_\beta = X - \{(-1,0)\}$ and $x_0 = (0,0) \in A_\alpha \cap A_\beta$. Let $\omega = [\gamma] \in \pi_1(A_\alpha \cap A_\beta, x_0)$ be a loop that winds around $(1,0)$.

The loop γ gives rise to different elements in each of the groups considered:

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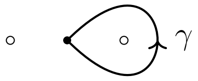
[3] This is not as obvious: it is not a priori clear that a homotopy in the disjoint union $\bigsqcup U_\alpha$ implies a homotopy in the quotient, see Section 2 in the "Additional Material for Weeks 5-6" document.



- $A_\alpha \cap A_\beta \simeq S^1 \vee S^1$ and the fundamental group $\pi_1(A_\alpha \cap A_\beta, x_0)$ is the free group on two generators a and b, with $[\gamma] = a$ one of them;
- $A_\alpha \simeq S^1$, and $(\iota_{\alpha\beta})_*([\gamma])$ is a generator of $\pi_1(A_\alpha, x_0) \cong \mathbb{Z}$;
- $A_\beta \simeq S^1$, but $(\iota_{\beta\alpha})_*([\gamma]) = e$, the constant loop in $\pi_1(A_\beta, x_0)$;
- $A_\alpha \cup A_\beta = \mathbb{R}^2$ and the image of γ under both $(\iota_\alpha)_* \circ (\iota_{\alpha\beta})_*$ and $(\iota_\beta)_* \circ (\iota_{\beta\alpha})_*$ is the unit element in the trivial group $\pi_1(\mathbb{R}^2, x_0)$.
- In $\pi_1(A_\alpha, x_0) * \pi_1(A_\beta, x_0)$, the concatenation of elements of $\pi_1(A_\alpha, x_0)$ with elements of $\pi_1(A_\beta, x_0)$ does not reduce, unless at least one of these is the unit element. In our case:

$$(\iota_{\alpha\beta})_*([\gamma]) \bullet (\iota_{\beta\alpha})_*([\gamma])^{-1} = (\iota_{\alpha\beta})_*([\gamma]) e_\beta = (\iota_{\alpha\beta})_*([\gamma]).$$

Note that even if the image of $[\gamma]$ in $\pi_1(A_\alpha, x_0)$ and in $\pi_1(A_\beta, x_0)$ "looks the same", we could still not cancel out concatenations of such elements in the free product $\pi_1(A_\alpha, x_0) * \pi_1(A_\beta, x_0)$, because as subgroups of this free product, these groups have only the empty word in common. In the free product, one can only concatenate elements from different groups, combining adjacent elements only if they come from the same group.

Lecture 19

Recall the setting of the Seifert-van Kampen theorem. Let $X = \bigcup_\alpha A_\alpha$ be an open cover and denote by $\iota_\alpha \colon A_\alpha \hookrightarrow X$ and $\iota_{\alpha\beta} \colon A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ the inclusion maps. Assume that $x_0 \in \bigcap_\alpha A_\alpha$. Then the inclusion maps induce maps between fundamental groups, as illustrated in the following commutative diagram:

$$\begin{tikzcd}
& \pi_1(A_\alpha, x_0) \arrow[dr] \arrow[drr, bend left, "(\iota_\alpha)_*"] & & \\
\pi_1(A_\alpha \cap A_\beta, x_0) \arrow[ur, "(\iota_{\alpha\beta})_*"] \arrow[dr, "(\iota_{\beta\alpha})_*"'] & & *_\alpha \pi_1(A_\alpha, x_0) \arrow[r, dashed, "\Phi"] & \pi_1(X, x_0) \\
& \pi_1(A_\beta, x_0) \arrow[ur] \arrow[urr, bend right, "(\iota_\beta)_*"'] & &
\end{tikzcd}$$

Explicitly, each element of $*_\alpha \pi_1(A_\alpha, x_0)$ is a reduced word $[\gamma_1] \cdots [\gamma_m]$, with $[\gamma_i] \in \pi_1(A_{\alpha_i}, x_0)$, no γ_i the trivial loop, and $\alpha_i \neq \alpha_{i+1}$ for $1 \leq i < m$. The induced map Φ is defined by

$$\Phi([\gamma_1] \cdots [\gamma_m]) = (\iota_{\alpha_1})_*([\gamma_1]) \bullet \cdots \bullet (\iota_{\alpha_m})_*([\gamma_m]) = [\gamma_1 * \cdots * \gamma_m],$$

where in the last line we consider γ_i as a loop in X (formally, $\iota_{\alpha_i} \circ \gamma_i$). Recall that the subgroup $N \leq *_\alpha \pi_1(A_\alpha, x_0)$ was defined as the normal subgroup generated by elements of the form $(\iota_{\alpha\beta})_*([\omega])(\iota_{\beta\alpha})_*([\omega])^{-1}$.

Theorem 19.1. *(Seifert-van Kampen) Let $X = \bigcup_\alpha A_\alpha$ be a cover with open sets and assume $x_0 \in \bigcap A_\alpha$. Then:*

***(I)** If for all α, β, $A_\alpha \cap A_\beta$ is path-connected, then the map*

$$\Phi = *_\alpha (\iota_\alpha)_* \colon *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

is surjective.

***(II)** If in addition for every α, β, γ the intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then* $\ker \Phi = N$, *and hence*

$$\pi_1(X, x_0) \cong *_\alpha \pi_1(A_\alpha, x_0)/N.$$

The first part is a consequence of the following lemma, which was already used in the derivation of the fundamental group of the sphere S^n for $n \geq 2$.

Lemma 19.2. *Let $X = \bigcup_\alpha A_\alpha$ be an open cover of a topological space, assume that $A_\alpha \cap A_\beta$ is path connected for all α, β and that $x_0 \in \bigcap A_\alpha$. Then every loop γ in X factors as*

$$[\gamma] = [\gamma_1] \bullet \cdots \bullet [\gamma_m],$$

with γ_i a loop in A_{α_i}.

Proof. Let $\gamma\colon I \to X$ be given, and consider the open cover $I = \bigcup_\alpha \gamma^{-1}(A_\alpha)$. Each of the $\gamma^{-1}(A_\alpha)$ is the union of open intervals, giving a cover of I by open intervals. By the Lebesgue Covering Lemma, there is a sequence

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that $\gamma([t_{i-1}, t_i]) \subset A_{\alpha_i}$ for some A_{α_i} and $1 \leq i \leq m$. In particular, for every end-point t_i we have that $\gamma(t_i) \in A_{\alpha_i} \cap A_{\alpha_{i+1}}$ for $1 \leq i < m$. It follows that for every $i \in \{1, \ldots, m-1\}$ there exists a path β_i from $\gamma(t_i)$ to x_0 in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$, with inverse path $\overline{\beta}_i$. We can then consider the modified path

$$\tilde{\gamma} = \gamma_1 * \cdots * \gamma_m,$$

where the γ_i are loops based at x_0, defined as

$$\gamma_i = \begin{cases} \gamma|_{[t_0,t_1]} * \beta_1 & \text{if } i = 1 \\ \overline{\beta}_{i-1} * \gamma|_{[t_{i-1},t_i]} * \beta_i & \text{if } i \in \{2, \ldots, m-1\} \\ \overline{\beta}_{m-1} * \gamma_{[t_{m-1},t_m]} & \text{if } i = m \end{cases}$$

Since the combinations $\beta_i * \overline{\beta}_i$ are the trivial loop, we have $\gamma \overset{\partial}{\simeq} \tilde{\gamma}$, and hence $[\gamma] = [\tilde{\gamma}] = [\gamma_1] \bullet \cdots \bullet [\gamma_m]$. □

Proof of Theorem 20.1 (I). For part (I), let $[\gamma] \in \pi_1(X, x_0)$. By Lemma 19.2, we can write $[\gamma] = [\gamma_1] \bullet \cdots \bullet [\gamma_m]$, with each γ_i a loop in one specific A_{α_i}. Moreover, we can assume that every γ_i is not the trivial loop, and that $\alpha_i \neq \alpha_{i+1}$ for $1 \leq i < m-1$ (otherwise we can just join γ_i and γ_{i+1} to one loop). This means that when considering $[\gamma_i]$ as elements of $\pi_1(A_{\alpha_i}, x_0)$, the word

$$[\gamma_1] \cdots [\gamma_m]$$

is a reduced word in $*_\alpha \pi_1(A_\alpha, x_0)$. By definition,

$$\Phi([\gamma_1] \cdots [\gamma_m]) = (\iota_{\alpha_1})_*([\gamma_1]) \bullet \cdots \bullet (\iota_{\alpha_m})_*([\gamma_m]) = [\gamma_1 * \cdots * \gamma_m] = [\gamma],$$

which shows that the map is surjective provided the pairwise intersections are path-connected. □

Lecture 20

Recall the setting of the Seifert-van Kampen theorem. Let $X = \bigcup_\alpha A_\alpha$ be an open cover and denote by $\iota_\alpha \colon A_\alpha \hookrightarrow X$ and $\iota_{\alpha\beta} \colon A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ the inclusion maps. Assume that $x_0 \in \bigcap_\alpha A_\alpha$. Then the inclusion maps induce maps between fundamental groups, as illustrated in the following commutative diagram:

$$\begin{tikzcd}
& \pi_1(A_\alpha, x_0) \arrow[dr] \arrow[drr, bend left, "(\iota_\alpha)_*"] & & \\
\pi_1(A_\alpha \cap A_\beta, x_0) \arrow[ur, "(\iota_{\alpha\beta})_*"] \arrow[dr, "(\iota_{\beta\alpha})_*"'] & & *_\alpha \pi_1(A_\alpha, x_0) \arrow[r, dashed, "\Phi"] & \pi_1(X, x_0) \\
& \pi_1(A_\beta, x_0) \arrow[ur] \arrow[urr, bend right, "(\iota_\beta)_*"'] & &
\end{tikzcd}$$

Explicitly, each element of $*_\alpha \pi_1(A_\alpha, x_0)$ is a reduced word $[\gamma_1] \cdots [\gamma_m]$, with $[\gamma_i] \in \pi_1(A_{\alpha_i}, x_0)$, no γ_i the trivial loop, and $\alpha_i \neq \alpha_{i+1}$ for $1 \leq i < m$. The induced map Φ is defined by

$$\Phi([\gamma_1] \cdots [\gamma_m]) = (\iota_{\alpha_1})_*([\gamma_1]) \bullet \cdots \bullet (\iota_{\alpha_m})_*([\gamma_m]) = [\gamma_1 * \cdots * \gamma_m],$$

where in the last line we consider γ_i as a loop in X (formally, $\iota_{\alpha_i} \circ \gamma_i$). Recall that the subgroup $N \leq *_\alpha \pi_1(A_\alpha, x_0)$ was defined as the normal subgroup generated by elements of the form $(\iota_{\alpha\beta})_*([\omega])(\iota_{\beta\alpha})_*([\omega])^{-1}$.

Theorem 20.1. *(Seifert-van Kampen) Let $X = \bigcup_\alpha A_\alpha$ be a cover with open sets and assume $x_0 \in \bigcap A_\alpha$. Then:*

(I) *If for all α, β, $A_\alpha \cap A_\beta$ is path-connected, then the map*

$$\Phi = *_\alpha (\iota_\alpha)_* \colon *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

is surjective.

(II) *If in addition for every α, β, γ the intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then* $\ker \Phi = N$*, and hence*

$$\pi_1(X, x_0) \cong *_\alpha \pi_1(A_\alpha, x_0)/N.$$

We outline a the proof of Part (II) of the Seifert-van Kampen Theorem. This follows from another lemma on the composition of loops. Assume again that $X = \bigcup_\alpha A_\alpha$ is an open cover with $x_0 \in \bigcap_\alpha A_\alpha$. Let $[f] \in \pi_1(X, x_0)$. A *factorization* of $[f]$ is a sequence

$$[f_1] \cdots [f_m]$$

such that $[f_i] \in \pi_1(A_{\alpha_i}, x_0)$ and $f \overset{\partial}{\simeq} f_1 * \cdots * f_m$. If $\alpha_i \neq \alpha_{i+1}$ for $1 \leq i < m$ and $[f_i] \neq e_{\alpha_i}$ for all i, then such a factorization simply gives a reduced word in $*_\alpha \pi_1(A_\alpha, x_0)$. We consider the following operations on such words:

- Reduction/expansion: If $[f_i], [f_{i+1}] \in \pi_1(A_\alpha, x_0)$, then

$$[f_1] \cdots [f_i][f_{i+1}] \cdots [f_m] \leftrightarrow [f_1] \cdots [f_i * f_{i+1}] \cdots [f_m]$$

- Exchange: If $[f_i] = (\iota_{\alpha\beta})_*([\omega])$ and $[g_i] = (\iota_{\beta\alpha})_*([\omega])$ for $[\omega] \in \pi_1(A_\alpha \cap A_\beta, x_0)$, then

$$[f_1] \cdots [f_i] \cdots [f_m] \leftrightarrow [f_1] \cdots [g_i] \cdots [f_m]$$

We call two factorization *equivalent* if they can be related a sequence of reductions, expansions or exchanges. Note that in contrast to the reduction of a word, we allow to exchange elements of $\pi_1(A_\alpha, x_0)$ with element from $\pi_1(A_\beta, x_0)$ that arise from the same element in $\pi_1(A_\alpha \cap A_\beta, x_0)$.

Lemma 20.2. *Any two factorizations $[f_1] \cdots [f_k]$ and $[f'_1] \cdots [f'_\ell]$ of $[f] \in \pi_1(X, x_0)$ are equivalent.*

Proof. Since $f \overset{\partial}{\simeq} f_1 * \cdots * f_k$ and $f \overset{\partial}{\simeq} f'_1 * \cdots * f'_\ell$, there exists a homotopy $G\colon I \times I \to X$ with $g_0 = f_1 * \cdots * f_k$ and $g_1 = f'_1 * \cdots * f'_\ell$. Using an approach similar to the proof of Part (I), we aim to decompose the homotopy by finding intermediate paths $\gamma_0, \ldots, \gamma_N\colon I \to X$ such that $\gamma_0 = g_0$, $\gamma_N = g_1$, and each γ_i has a factorization in such a way that the factorization of γ_{i+1} arises from that of γ_i by a reduction, expansion, or exchange operation.

(1) We first decompose $I \times I$ into rectangles. Consider the open cover $I \times I \subset \bigcup_\alpha G^{-1}(A_\alpha)$. By the product topology on the square, every open set $G^{-1}(A_\alpha)$ can be written as the union of open rectangles, and since $I \times I$ is compact, we have finitely many such rectangles. The closure of these rectangles covers $I \times I$, and after a common refinement (or using the Lebesgue Covering Lemma, Section 6 in Week 3-4 Additional Material), we can assume that we have a decomposition $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for each such rectangle $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$ there exists an α with $G([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subset A_\alpha$. Since $G^{-1}(A_\alpha)$ is open, there exists an $\epsilon > 0$ such that the small horizontal displacement $[s_i + \epsilon, s_{i+1} + \epsilon] \times [t_j, t_{j+1}]$ remains in $G^{-1}(A_\alpha)$. By shifting the rectangles in this way to the left or right, we can ensure that no point lies in more than three rectangles (see Figure 20.24).

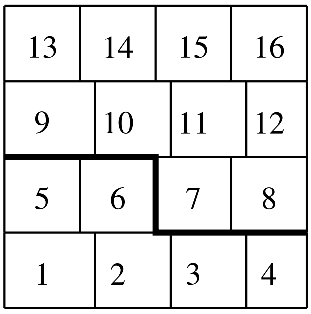
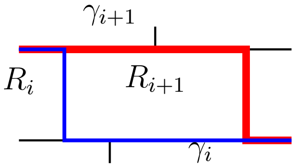


Figure 20.24: The subdivision of $I \times I$ and the path γ_6



(2) We next define paths along the rectangles. Let γ_i be the path from the left boundary to the right boundary that separates the rectangles $R_1, \dots, R_i$ from $R_{i+1}, \dots, R_N$. In particular, γ_0 is the lower boundary and γ_N the upper boundary, and γ_i and γ_{i+1} only differ on the boundary of R_{i+1} (γ_i goes under it, and γ_{i+1} above it). The homotopy from g_0 to g_1 gives a homotopy from γ_i to γ_{i+1}, by "pushing γ_i across R_{i+1}", i.e., applying G to a homotopy from the left-and-lower boundary of R_{i+1} to the right-and-upper boundary.

(3) We next associate a loop to every edge of a rectangle. Since every vertex v is in the intersection of at most three rectangles, it has the property that $G(v) \in A_\alpha \cap A_\beta \cap A_\gamma$ for some α, β, γ. By the assumption of path-connectedness, there exists a path h_v from $G(v)$ to x_0 in $A_\alpha \cap A_\beta \cap A_\gamma$. If R_i and R_{i+1} are adjacent rectangles with $G(R_i) \subset A_\alpha$ and $G(R_{i+1}) \subset A_\beta$, then their common boundary μ defines a path $G \circ \mu \in A_\alpha \cap A_\beta$ with endpoints v and w, and also a loop $\overline{h}_v * (G \circ \mu) * h_w \subset A_\alpha \cap A_\beta$. If follows that every γ_i factors as a product of loops. Note however that the factorizations of γ_i and γ_{i+1} have common loops ω, but that these give rise to different elements $[\omega]$, depending on which fundamental group these classes are taken in.

(4) We can now move from a factorization of γ_i to one of γ_{i+1} as follows. Let $[\omega] \in \pi_1(A_\alpha, x_0)$ be an element in the factorization of γ_i that arises from a loop that also corresponds to a boundary of R_{i+1}. Replace this element with the corresponding class in $\pi_1(A_\beta, x_0)$ (exchange operation). Then replace the loops corresponding to the left and lower boundary with loops corresponding to the upper and right boundaries (reduction, replacement by homotopic loop, and expansion).

(5) The "boundary cases" may need to be treated separately. Altogether, we see that we can get from a factorization of g_0 to a factorization of g_1 by a sequence of exchanges, reductions, and expansions, thus showing that homotopic loops are equivalent in this sense. □

Proof of Seifert-van Kampen, Part (II). Let $w = [f_1] \cdots [f_m]$ be a reduced word in $*_\alpha \pi_1(A_\alpha, x_0)$. If $[\omega] \in \pi_1(A_\alpha \cap A_\beta, x_0)$ and $[f_i] = (\iota_{\alpha\beta})_*([\omega])$ and $[g_i] = (\iota_{\beta\alpha})_*([\omega])$, then replacing $[f_i]$ with $[g_i]$ (performing an exchange operation, and possibly reducing if necessary) gives a word v such that $w \bullet v^{-1} \in N$. The same is true if the word is expanded before the exchange and reduced after it. Therefore, if $w = [f_1] \cdots [f_k]$ and $v = [f_1'] \cdots [f_\ell']$ are two elements in $*_\alpha \pi_1(A_\alpha, x_0)$ that arise from each other by exchanges, expansions and reductions, we have $w \bullet v^{-1} \in N$.

If $\Phi(w) = [e]$, then $e \overset{\partial}{\simeq} f_1 * \cdots * f_k$, so that w constitutes a factorization of e. Moreover, this factorization is equivalent to the empty factorization ϵ in the sense that they can be transformed into one another by a sequence of exchanges, expansions and reductions. It follows that $w \in N$, and hence $\ker \Phi \subset N$. The other inclusion is easy. □

Lecture 21

In this lecture we discuss a few applications of the Seifert-van Kampen Theorem. We then introduce the notion of CW complex that allows to describe topological spaces more effectively.

Example 21.1. Let $X = \bigvee_{n\in\mathbb{N}} C_n$ be the wedge product of pointed topological spaces $(C_n, x_0 = (0,0))$, where each C_n is a circle centered at $(1/n, 0)$ of radius $1/n$ in $\mathbb{R}^2$. Each such circle is homeomorphic to S^1. In Lecture 18 we saw that

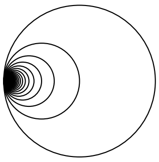
$$\pi_1\left(\bigvee_{n\in\mathbb{N}} C_n, x_0\right) \cong *_\alpha \pi_1(S^1, x_0),$$

which is isomorphic to the free group generated by copies of $\mathbb{Z}$. Using the fact that products of countable sets are countable, and countable unions of countable sets are countable, it can be shown that the fundamental group $\pi_1(X, x_0)$ is countable.

Example 21.2. Consider now the "Hawaiian earrings", defined as

$$X = \bigcup_{n\in\mathbb{N}} C_n,$$

with the *subspace topology* induced from $\mathbb{R}^2$.



One might think that this set is the same as an infinite wedge product of Example 21.1, but we will see that this is not the case. To see this, consider, for each $n \in \mathbb{N}$, the retraction $r_n\colon X \to C_n$ that maps all C_i, $i \neq n$, to $(0,0)$ and is the identity on C_n. This induces a sequence of surjective maps

$$(r_n)_*\colon \pi_1(X, x_0) \to \pi_1(C_n, x_0) \cong \mathbb{Z}.$$

The product of these maps gives rise to a surjective map

$$\begin{aligned}\rho\colon \pi_1(X, x_0) &\to \Pi_{n\geq 1}\mathbb{Z},\\ [\omega] &\mapsto ((r_1)_*([\omega]), (r_2)_*([\omega]), \dots),\end{aligned}$$

where we identified elements in $\pi_1(C_n, x_0)$ with their image in $\mathbb{Z}$. This map is surjective, since for every sequence of integers $\{k_n\}_{n\in\mathbb{N}}$ we can construct a loop in $\pi_1(X, x_0)$ that wraps around C_n k_n times in the time interval $[1 - 1/n, 1 - 1/(n + 1)]$ and whose image is $\{k_n\}_{n\in\mathbb{N}}$. Such a loop $\varphi\colon I \to X$ is clearly continuous in each interval $[1 - 1/n, 1 - 1/(n + 1)]$. It is also continuous at 1, since every open neighbourhood U of $x_0 \in X$ contains all but finitely many circles C_n, and hence its preimage $\varphi^{-1}(U)$ is the complement of a finite union of closed sets, and as such is open. The product $\Pi_{n\geq 1}\mathbb{Z}$ is uncountable (we can represent every real number as an infinite sequence of integers), so that $\pi_1(X, x_0)$ also has to be uncountable. The fundamental group of the wedge product of the C_n, however, is the free product of the groups $\pi_1(C_n, x_0)$. This is the set of all *finite* (reduced) words that can be assembled from elements in the individual groups, and this set is *countable* (prove this!). The fundamental group $\pi_1(X, x_0)$ turns out to be rather complicated.

Exercise 21.3. Find where the construction of a surjection $\pi_1(X, x_0) \to \Pi_{n\geq 1}\mathbb{Z}$ in Example 21.2 breaks down if we define $X = \bigvee_{n\geq 1} C_n$ instead of taking the union.

Example 21.4. In some areas of topology, such as knot theory, one is interested in the *complement* of certain sets in $\mathbb{R}^3$. Let $S^1 \subset \mathbb{R}^3$. Then one can show that $X = \mathbb{R}^3 - S^1$ is homotopy equivalent to a wedge product $S^1 \vee S^2$. The Seifert-van Kamplen Theorem then implies that $\pi_1(X, x_0) \cong \mathbb{Z}$. If $X = \mathbb{R}^3 - (A \cup B)$, where $A \cong S^1$ and $B \cong S^1$ are two circles, then the fundamental group of X differs depending on whether the circles are linked or not! For example, if the circles are not linked, then one can show that X is homotopy equivalent to $S^2 \vee S^2 \vee S^1 \vee S^1$, and hence $\pi_1(X, x_0) \cong \mathbb{Z} * \mathbb{Z}$. If, on the other hand, the circles are *linked*, then X is homotopy equivalent to $S^2 \vee (S^1 \times S^1)$, the wedge product of a sphere and a torus, and hence Seifert-van Kampen implies that $\pi_1(X, x_0) \cong \mathbb{Z} \times \mathbb{Z}$.

To be able to compute with, and compare, common topological spaces more effectively, we introduce the concept of a CW complex. CW complexes are topological spaces that can be assembled from simpler spaces by “glueing” cells together. Many, but not all, interesting topological spaces have the structure of a CW complex.

21.1 CW complexes

Definition 21.5. A **CW complex** is a topological space X that is built up inductively as follows.

1. The **zero-skeleton** X^0 is a discrete set;

2. Given X^{n-1}, a collection of closed *disks* $\{D^n_\alpha\}$ with $D^n_\alpha \cong B^n$, and $S^{n-1}_\alpha = \partial D^n_\alpha$, with **attaching maps**

$$\varphi_\alpha\colon S^{n-1}_\alpha \to X^{n-1},$$

define

$$X^n = (X^{n-1} \sqcup \bigsqcup_\alpha D^n_\alpha)/\sim,$$

where $\sim$ is the equivalence relation $x \sim \varphi_\alpha(x)$ for all $x \in S^{n-1}_\alpha$.

3. Define $X = \bigcup_n X^n$, equipped with the **weak topology**: a set $A \subset X$ is open if and only if $A \cap X^n$ is open *in* X^n for every n.

The disks D^n_α are called n-cells, and their interiors $e^n_\alpha = D^n_\alpha - S^{n-1}_\alpha$ are the open n-cells. The set X^n is called the n-skeleton of the CW complex. A CW complex is called **finite-dimensional** if $X = X^n$ for some n, and the largest n for which there are cells in the complex is called the **dimension** of the complex. A CW complex is called **finite** if it has only finitely many cells.

Example 21.6. A one-dimensional CW complex is called a (topological) **graph**. It consists of X^0 (the vertices), with X^1 arising by attaching the endpoints of intervals D^1_α to the vertices.

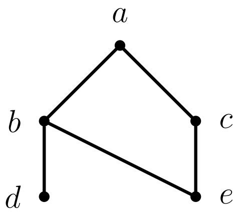


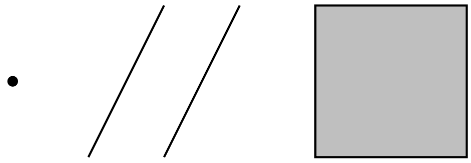
Figure 21.25: A graph.

A graph need not be finite. We can take, for example, as nodes $X^0 = \mathbb{Z}$, as edges copies of the unit interval I, and attaching maps φ_n defined by $\varphi_n(0) = n$ and $\varphi_n(1) = n + 1$. The resulting CW complex is homeomorphic to $\mathbb{R}$.

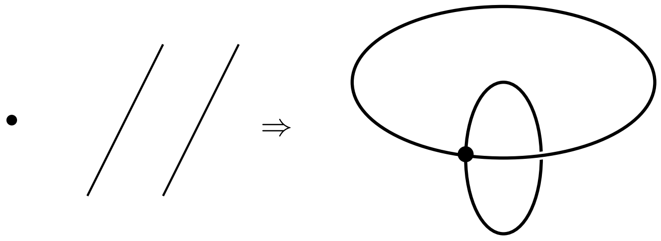
Exercise 21.7. Show that every connected graph is homeomorphic to a wedge of spheres $\vee_\alpha S^1$.

Example 21.8. We can fill in some of the closed areas of a graph, which gives rise to a two-dimensional CW complex. Other examples are polyhedra (the cube, the tetrahedron, etc.). The space $\mathbb{R}^n$ can be expressed as a CW complex in many different ways. The CW structure of a topological space is clearly not unique.

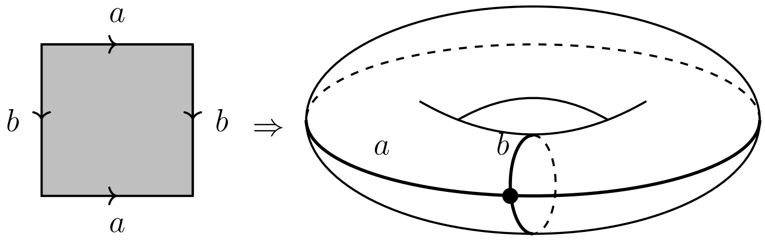
Example 21.9. The **torus** is an example of a two-dimensional CW complex. The ingredients are: one point $X^0 = \{x\}$, two line segments $\{I_1, I_2\}$, and one square D^2 (homeomorphic to the ball B^2).



For each of the line segments to the point by means of a map $\varphi_i \colon \partial I \to X^0$ (there is only one way of doing this).



We then attach the square to the resulting graph by a map $\varphi\colon \partial D^2 \to X^1$, mapping the upper and lower boundaries to one circle, and the left and right boundaries to the other circle. This is often visualized by drawing the square and labelling the edges in a way that indicates which edges are identified in which way:



Exercise 21.10. Show that the torus defined in this way is homeomorphic to $\mathbb{T}^2 = S^1 \times S^1$.

Exercise 21.11. Show that the sphere S^n is a CW complex with one 0-cell and one n-cell.

Lecture 22

Recall the definition of a CW complex. We will discuss a few interesting examples of CW complexes and see how to compute the fundamental group using the Seifert-van Kampen Theorem.

22.1 The Möbius strip and projective space

So far we have basic examples, such as graphs, the torus, and the sphere S^n. In this section we will revisit the projective plane $\mathbb{RP}^2$, and show that it can be characterized by glueing a disk to the boundary of a Möbius strip. We will then use this characterization as an alternative way of computing the fundamental group of $\mathbb{RP}^2$.

Example 22.1. The Möbius strip M can be defined as $I \times I$ by identifying $(0, x)$ with $(1, 1-x)$ for $x \in I$.

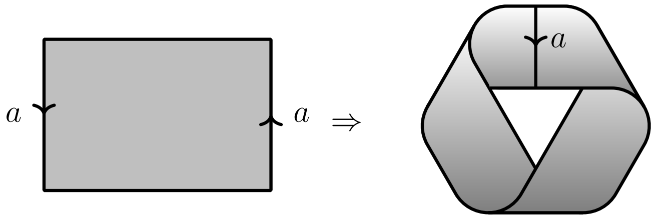


Figure 22.26: The Möbius strip

There is one obvious CW complex structure on the Möbius strip: take 0 cells (the end points of a), three 1-cells (the line segment a and the upper and lower boundaries of the rectangle), and one 2-cell, a rectangle itself. This is not the only way to describe the Möbius strip.

The Möbius strip has a circle at its centre, namely the image of $I \times \{1/2\}$ (since $(0, 1/2) \sim (1, 1/2)$). The Möbius strip deformation retracts to this circle by taking the homotopy on the rectangle,

$$\tilde{F} \colon (I \times I) \times I \to I \times I, \quad ((x, y), t) \mapsto (x, (1-t)(y - 1/2) + 1/2).$$

Since $1 - [(1-t)(y-1/2) + 1/2] = (1-t)(1-y-1/2) + 1/2$, the homotopy carries over to a homotopy in the quotient. It follows that $\pi_1(M) \cong \mathbb{Z}$. The Möbius strip also has *only one* circle at its boundary, the image of $(I \times \{0\}) \times (I \times \{1\})$ under the quotient map.

Example 22.2. Real projective space $\mathbb{RP}^n$. Recall that

$$\mathbb{RP}^n = S^n/(x \sim -x),$$

the n-sphere with antipodal points identified (equivalently: the set of lines, that is, $\mathbb{R}^{n+1}$ with $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{R}$). Let $q\colon S^n \to \mathbb{RP}^n$, $x \mapsto [x]$, be the quotient map. We can define a CW structure on $\mathbb{RP}^n$ recursively as follows. Consider the open set $U_0 = \{[(x_0, x_1, \dots, x_n)]\colon x_0 \neq 0\}$. The set $\mathbb{RP}^n - U_0 = \{[(0, x_1, \dots, x_n)]\}$ is homeomorphic to $\mathbb{RP}^{n-1}$. Moreover, since q is a two-sheeted covering map, and the preimage $q^{-1}(U_0)$ consists of the disjoint union of the sets $\{x_0 > 0\}$ and $\{x_0 < 0\}$, each of which is the interior of an n-ball that maps homeomorphically to U_0, and hence $U_0 \cong e^n$, an open disk. Setting $D^n = \{x_0 \geq 0\}$ and $S^{n-1} = \partial D^n = \{(0, x_1, \dots, x_n)\}$, we get the two-fold covering

$$\varphi\colon S^{n-1} \to \mathbb{RP}^{n-1}$$

as attaching map (where we identified $\mathbb{RP}^{n-1} = \mathbb{RP}^n - U_0$), with $\mathbb{RP}^n$ arising as

$$\mathbb{RP}^{n-1} \sqcup D^n/(x \sim \varphi(x)).$$

We can continue this process recursively with $\mathbb{RP}^{n-1}$. As each step adds one open n-cell to the construction, we get a characterization of real projective space as

$$\mathbb{RP}^n = \{pt\} \cup e^1 \cup e^2 \cup \cdots \cup e^n,$$

with one open n-cell in each dimension.

In low dimensions, we have $\mathbb{RP}^0 = \{\mathrm{pt}\}$, $\mathbb{RP}^1 = \mathbb{RP}^0 \sqcup D^1/(0 \sim 1)$, which characterizes $\mathbb{RP}^1$ as a circle. For $\mathbb{RP}^2$, we attach a 2-cell by taking a disk D^2 and attaching the boundary circle S^1 to $\mathbb{RP}^1$ via the two fold covering $S^1 \to \mathbb{RP}^1$.

One way of thinking about $\mathbb{RP}^2$ is to take the closed upper hemisphere of a sphere S^2. Each point there corresponds to a a unique point in $\mathbb{RP}^2$, except at the boundary, where we have to identify antipodal points. But this makes the boundary an $\mathbb{RP}^1$. One can visualize the cell decomposition of $\mathbb{RP}^2$ as follows:

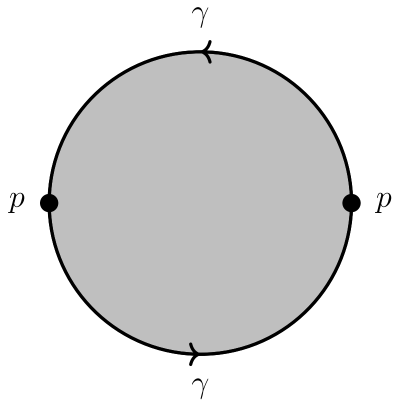
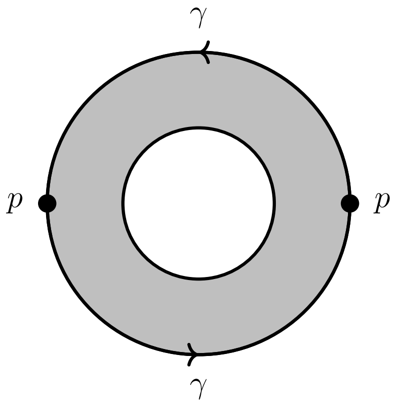


Figure 22.27: Cell decomposition of $\mathbb{RP}^2$.

The figure shows a 2-dimensional disk whose boundary disk is subdivided into cells that are identifies (the lines being identified along the arrow direction).

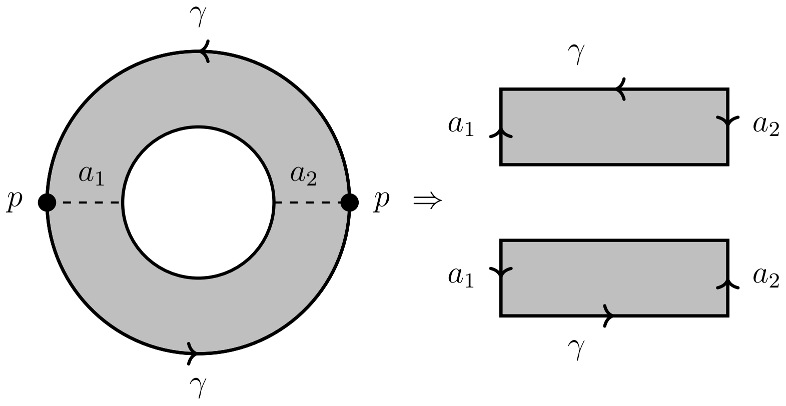
Example 22.3. ($\mathbb{RP}^2$ meets the Möbius strip). Consider the cell decomposition of $\mathbb{RP}^2$ as given in Figure 22.27, and let X be the space obtained by removing a closed disk from the interior of $\mathbb{RP}^2$.


γ
p
p
γ

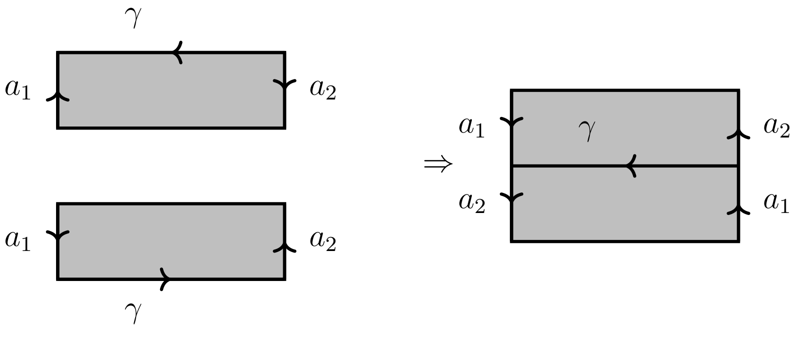
Formally, we can describe X as

$$X = S^1 \times I/(x,1) \sim (-x,1),$$

as $S^1 \times I$ describes the annulus, and the identification simply identifies antipodal points on one boundary of the annulus, but not on both. We claim that $X \cong M$, the Möbius strip. Visually, this can be seen by first "detaching" the annulus (keeping track of where the identifications happen),


γ
a_1
a_2
p
p
$\Rightarrow$
γ
γ
a_1
a_2
a_1
a_2
γ

and then "reattaching" along γ, where we flip the upper rectangle around and rotate the lower rectangle by 180 degrees:


γ
a_1
a_2
a_1
a_2
γ
$\Rightarrow$
a_1
γ
a_2
a_2
a_1

If we denote the concatenation $a = a_1 * a_2$, then we get exactly the characterization of Figure 22.26, with γ the circle at the centre. As a consequence of this example, we see that we can obtain the projective plane by glueing a 2-cell D^2 to the boundary of a Möbius strip.

Exercise 22.4. Describe the homeomorphism $X \to M$ described above explicitly.

Given the above examples, we can compute the fundamental group of $\mathbb{RP}^2$ as follows. Recall the characterization of of $\mathbb{RP}^2$ from Figure 22.27, and denote by e^2 the interior of the disk. Consider a cover of $\mathbb{RP}^2$ as follows. Consider an open disk $B \subset e^2$ in $\mathbb{RP}^2$ and a closed disk $C \subset B$, and define $A = \mathbb{RP}^2 - C$ (see Figure 22.28). Then $\mathbb{RP}^2 = A \cup B$.

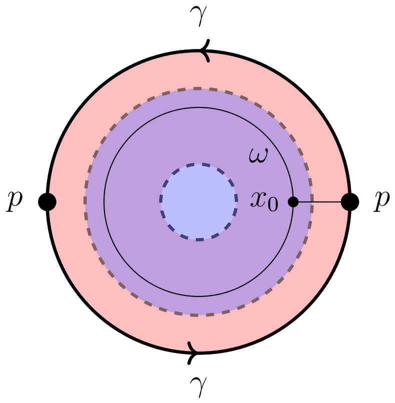


Figure 22.28: An open cover of $\mathbb{RP}^2$

Fix a base point $x_0 \in A \cap B$. Clearly, $\pi_1(B, x_0) = \mathbf{1}$, the trivial group, since B is just an open disk. The intersection $A \cap B$ is homotopic to a circle, represented by a loop ω, so that $\pi_1(A \cap B, x_0) = \langle [\omega] \rangle \cong \mathbb{Z}$. The set A, in turn, is the interior of a Möbius strip, as seen in Example 22.3, with γ representing the inner circle. As seen in Example 22.1, A deformation retracts to γ (or, more precisely, to a circle homotopic to γ but with basepoint x_0, see the figure), so that $\pi_1(A, x_0) \cong \langle [\gamma] \rangle \cong \mathbb{Z}$.

Since the fundamental group $\pi_1(B, x_0)$ is trivial, the free group $\pi_1(A, x_0) * \pi_1(B, x_0)$ is generated by $[\gamma]$. To get the fundamental group of $\mathbb{RP}^2$ using Seifert-van Kampen, we have to factor out elements that are multiples of

$$(\iota_{A\cap B})_*([\omega]),$$

where $\iota_{A\cap B}$ is the inclusion of $A \cap B$ in A. We can think of ω as the outer circle of a Möbius strip, and γ as the inner circle. Going around ω once corresponds to going around γ twice, so that

$$(\iota_{A\cap B})_*([\omega]) = [\gamma]^2.$$

By the Seifert-van Kampen Theorem,

$$\pi_1(\mathbb{RP}^2, x_0) \cong \langle [\gamma] \rangle / \langle [\gamma]^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

Lecture 23

45.2 Properties of CW complexes

Recall that we denoted closed cells of dimension n by D^n_α, their boundary by S^{n-1}_α, and $e^n_\alpha = D^n_\alpha - S^{n-1}_\alpha$ the open cell. To simplify notation, we will call the 0-cells, the points in X^0, e^0_α.

Given a CW complex X, for every closed cell D^n_α we have an inclusion map into the disjoint union of X^{n-1} with n-cells, that gives rise to a map into $X^n \subset X$ by applying the quotient map q to it,

$$\begin{tikzcd}
& X^{n-1} \sqcup \bigsqcup_\alpha D^n_\alpha \arrow[d, "q"] & \\
D^n_\alpha \arrow[ur, hook, "\iota"] \arrow[r] & X^n \arrow[r, hook] & X
\end{tikzcd}$$

where X^n arises by identifying $x \sim \varphi^n_\alpha(x)$ for points $x \in S^{n-1}_\alpha$. This gives rise to a **characteristic map**

$$\Phi^n_\alpha \colon D^n_\alpha \to X,$$

and the restriction of Φ^n_α to e^n_α is a homeomorphism to its image, also denoted by e^n_α. We can therefore characterize a CW complex as disjoint union of cells e^n_α.

Exercise 45.5. Show that the weak topology on X can be characterized by saying that $A \subset X$ is closed if and only if for each n, α, $(\Phi^n_\alpha)^{-1}(A)$ is closed in D^n_α.

Definition 45.6. A **subcomplex** of a CW complex X is a space A that is a union of cells e^α_n in X such that for every cell it also contains its closure.

We now study some important properties of CW complexes.

Proposition 45.7. *A compact topological subspace of a CW complex X is contained in a finite subcomplex.*

Proof. Let $C \subset X$ be a compact set, and assume that C intersects infinitely many cells e^n_α. Then there exists a sequence of points $S = \{x_1, x_2, \dots\} \subset C$ so that each x_i lies in a different cell. Using the characterization of the weak topology via the characteristic map one can show that S is closed in X. Moreover, as every subset of S is closed, the topology on S is the discrete topology. As a closed subset of C, S is compact, but any compact set in the discrete topology is finite, so S is finite. It follows that C is contained in finitely many cells. It remains to show that that a finite union of cells is contained in a finite subcomplex. This can be seen by induction on n. The statement is clearly true for $n = 0$, since a finite union of 0-cells is just a finite set of points. If $n \geq 1$, then for every e^n_α, the image of the attaching map $\varphi^n_\alpha \colon S^{n-1}_\alpha \to X^{n-1}$ is compact, hence contained in finite union of cells of dimension at most $n-1$, which by induction hypothesis are contained in a finite subcomplex A. Attaching D^n_α to this subcomplex gives a finite complex containing e^n_α. $\square$

The letter 'C' in CW complex means **closure finiteness**: the closure of every open cell meets only finitely many other cells. The 'W' stands for weak topology.

Definition 45.8. A topological space is called **normal** if any two disjoint closed subsets have disjoint open neighbourhoods. A topological spaces is called a **Hausdorff space**, if any two distinct points have disjoint open neighbourhoods.

Proposition 45.9. *A CW complex is normal (and hence Hausdorff).*

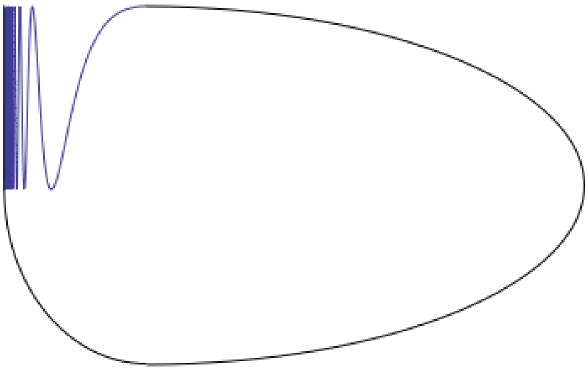
Definition 45.10. A topological space is called **locally contractible** if for every x and open neighbourhood U with $x \in U \subset X$ there exists an open set V with $x \in V \subset U$ such that V is contractible.

Example 45.11. Any open subset of $\mathbb{R}^n$ is contractible.

Example 45.12. Consider the Warsaw circle, defined as

$$W = \{(x, \sin(1/x) \colon x \in (0,1]\} \cup (\{0\} \times [-1,1]) \cup L,$$

where L is a curve jointing the first to sets in the union, with the subspace topology.



The Warsaw circle is not locally contractible. If it were, then it would be locally path connected (if V is a contractible neighbourhood of a point x, then any two points in V can be connected by a path via the homotopy between the identity on V and the retraction to a point in V). It is, however, not locally path connected. To see this, take any point on the piece $\{0\} \times I$, say $x = (0,0)$. Then every open neighbourhood of x of diameter less than 1 has infinitely many disconnected points. More precisely, if $V = \{y \in W \colon \|y\| < \varepsilon\}$ for $\varepsilon < 1$, then the points $(1/n\pi, 0)$ for integers $n > 1/\varepsilon\pi$ are all in V, but are not connected by a path. Note however that W is path-connected!

Proposition 45.13. *CW complexes are **locally contractible**.*

Corollary 45.14. *The Warsaw circle is not a CW complex.*

Lecture 24

Proposition 24.1. *If $A \subset X$ is a subcomplex of a CW complex X, then there exists an open set $U \subset X$ with $A \subset U$, and such that U deformation retracts to A.*

An important application is that we can apply the Seifert-van Kampen Theorem to decompositions $X = A \cup B$ into subcomplexes A and B such that $A \cap B$ is again a subcomplex. For example, if $A \subset U$ and $B \subset V$, then $\pi_1(U, x_0) = \pi_1(A, x_0)$, $\pi_1(V, x_0) = \pi_1(B, x_0)$, and $\pi_1(U \cap V, x_0) = \pi_1(A \cap B, x_0)$.

In the following, we will derive an important property of the fundamental group of CW complexes, namely that it depends only on the 2-skeleton! While we can derive this as a consequence of Proposition 24.1, give outline a proof from scratch, based on the Seifert-van Kampen Theorem.

Theorem 24.2. *For a path-connected CW-complex X with $x_0 \in X^2$, the inclusion $X^2 \hookrightarrow X$ induces an isomorphism of fundamental groups $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$.*

The statement can be interpreted intuitively as saying that by studying loops, we cannot distinguish higher-dimensional topological properties. Recall, for examples, the fundamental groups of the spheres and of projective spaces:

$$\pi_1(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & \text{for } n = 1, \\ \mathbb{Z}/2\mathbb{Z} & \text{for } n \geq 2 \end{cases}, \quad \pi_1(S^n) = \begin{cases} \mathbb{Z} & \text{for } n = 1, \\ \mathbf{1} & \text{for } n \geq 2 \end{cases}$$

That is, the fundamental group does not give us more information on higher-dimensional spheres other than that any loop on it is null-homotopic. There are various ways to get higher-dimensional information. One could study higher-dimensional **homotopy groups**, arising by considering maps $S^n \to X$ instead of loops (which can be considered as maps $S^1 \to X$). A different approach is via **homology** and **cohomology**, which is the subject of more advanced courses in algebraic topology.

We begin with an observation. Note that if X is a topological space and $\varphi_\alpha \colon S^1_\alpha \to X$ is a map that attaches a 2-cell D^2_α to X, then φ_α defines a loop $f_\alpha \colon I \to X$ on X based at $\varphi_\alpha(1)$ by setting $f_\alpha(t) = \varphi_\alpha(\exp(2\pi i t))$. While this loop may not be null-homotopic in X, it is null-homotopic in

$$Y := X \sqcup D^2_\alpha / (x \sim \varphi_\alpha(x)),$$

after attaching the cell. If X is path-connected, we can choose a basepoint $x_0 \in X$ and a path $h \colon I \to X$ with $h_\alpha(0) = x_0$, $h_\alpha(1) = \varphi_\alpha(1)$, and thus get a loop $\gamma_\alpha = h_\alpha * f_\alpha * \overline{h}_\alpha$. In this way, every attaching map gives rise to a loop in Y. The inclusion $X \hookrightarrow Y$ gives rise to a map of fundamental groups $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, and the class of every such loop, $[\gamma_\alpha]$, is contained in the kernel of this map.

Proposition 24.3. *Let X be a path-connected topological space and for fixed n, let $\varphi^n_\alpha \colon S^{n-1}_\alpha \to X$ be a collection of attaching maps, and set*

$$Y = X \sqcup \bigsqcup_\alpha D^n_\alpha / (x \sim \varphi^n_\alpha(x)).$$

Let $x_0 \in X$ be a point. Then

- *If $n = 2$, then*
$$\pi_1(Y, x_0) \cong \pi_1(X, x_0)/N,$$
where N is the normal subgroup generated by $[\gamma_\alpha]$, as defined above.

- *If $n > 2$, then*
$$\pi_1(Y, x_0) \cong \pi_1(X, x_0).$$

Proof. (Sketch) The proof is an application of the Seifert-van Kampen Theorem to a space $\tilde{Y}$ that deformation retracts to Y. Specifically, for each attached cell D^n_α consider a square $S_\alpha = I \times I$, a small path segment $p_\alpha \colon I \to D^n_\alpha$ with $p_\alpha(0) = h_\alpha(1) \in S^{n-1}_\alpha$ and $p_\alpha(1) \in e^n_\alpha$, and a map

$$\mu_\alpha \colon (I \times \{0\}) \cup (\{1\} \times I) \to Y, \quad (x, 0) \mapsto h_\alpha(x), \quad (1, y) \mapsto p_\alpha(y).$$

Define $\tilde{Y} = Y \sqcup \bigsqcup S_\alpha / \sim$, where the relation $\sim$ is defined by setting $x \sim \mu_\alpha(x)$ if x is in the lower-and-right boundary of S_α, and $(0, y) \sim (0, y')$ if $(0, y) \in S_\alpha$ and $(0, y') \in S_\beta$. The effect of this operation is to "lengthen" the paths from the base-point x_0 to the cells by turning them into stripes. The deformation retract of the rectangle to the lower boundary $I \times \{0\}$ induces a deformation retract of $\tilde{Y}$ to Y.

Choose points y_α in each cell e^n_α (and such that they do not lie on the path p_α). We now define the following subsets of $\tilde{Y}$:

- $A = \tilde{Y} - \bigcup_\alpha \{y_\alpha\}$;
- $B = \tilde{Y} - X$.

Since B consists of the cells e^n_α with the attached paths, it is contractible and we have $\pi_1(B, x_0) = \mathbf{1}$. By the homotopy that retracts the interior of a ball B^n without a point to its boundary, we see that $A \simeq X$. It follows that

$$\pi_1(Y, x_0) \cong \pi_1(\tilde{Y}, x_0) \cong \pi_1(X, x_0)/N,$$

where N is the normal subgroup generated by the images in $\pi_1(A, x_0)$ of elements of $\pi_1(A \cap B, x_0)$. If $n > 2$, then the cells D^n_α without a point y_α are still contractible, so $A \cap B$ is contractible and $\pi_1(A \cap B, x_0) = \mathbf{1}$, from which the claim follows in this case. In the case $n = 2$, one gets a loop for every attached cell D^2_α that is homotopic to a loop γ_α (after a basepoint change, where the original basepoint $x_0 \in X$ is moved up the line segment to a basepoint that is in $A \cap B$). $\square$

Proof. (of Theorem 24.2) If X is a finite-dimensional CW complex, then the statement follows from proposition 24.3 by induction: $X = X^n$ is constructed from X^{n-1} by attaching n-cells, and Proposition 24.3 tells us that this process does not alter the fundamental group if $n > 2$. If X is not finite-dimensional, we can still apply the proposition by noting that a loop γ in X is a compact subset, and therefore contained in a finite subcomplex in some X^n. Since $\pi_1(X^2, x_0) \cong \pi_1(X^n, x_0)$, every such loop is homotopic to a loop in X^2, and therefore the map $\pi_1(X^2, x_0) \to \pi_1(X, x_0)$ is surjective. To see that this is injective, let γ be a loop in X^2 that is homotopic, in X, to the constant loop via a homotopy $F\colon I \times I \to X$. As the image of F in X is compact, it is contained in a finite subcomplex X^n, and we can assume that $n > 2$. If follows that $[\gamma] = 0$ in $\pi_1(X^n, x_0)$, and we can use the injectivity of $\pi_1(X^2, x_0) \to \pi_1(X^n, x_0)$ to conclude that γ is null-homotopic in X^2. □

Note that we can get this result as a consequence of Proposition 24.1. For this, consider $X = X^n$, $A = X^{n-1}$ and $B = \bigcup_\alpha \Phi_\alpha(D^n_\alpha)$. Then $A \cap B = \bigcup_\alpha \Phi_\alpha(S^{n-1}_\alpha)$. Applying the Seifert-van Kampen Theorem to this CW decomposition, and using the fact that $\pi_1(B) = \mathbf{1}$, we get

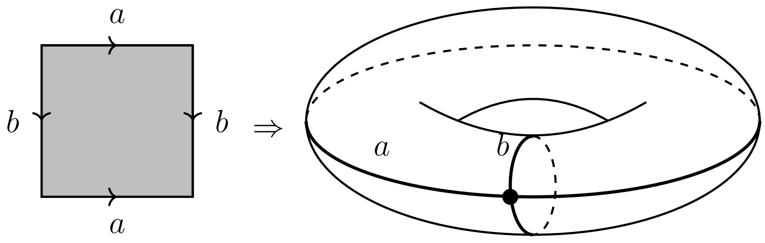
$$\pi_1(X^n) \cong \pi_1(X^{n-1})/N,$$

with N the normal subgroup generated by loops coming from $A \cap B$. Any such loop is in $\Phi_\alpha(S^{n-1}_\alpha)$, and therefore null-homotopic if $n > 2$, but not necessarily if $n = 2$.

Lecture 25

25.1 Generators and relations

In this lecture we introduce a way of describing groups using generators and relations, and how to interpret these in the case of the fundamental group of a topological space. We begin by illustrating an example: the torus. Recall that the torus, $\mathbb{T}^2 = S^1 \times S^1$, has fundamental group isomorphic to $\mathbb{Z} \times \mathbb{Z}$. This was derived by noting that the fundamental group of a product is the product of fundamental groups. We now discuss a different way of describing this fundamental group, that gives more insight into the topology of the problem. The starting point is the characterization of the torus as a rectangle with opposite sides identified by gluing them together.



In this characterization, the torus is defined as

$$\mathbb{T}^2 = I \times I / \sim,$$

with $(s,0) \sim (s,1)$ and $(0,t) \sim (1,t)$. Let $p = (1/2, 1/2)$ be the centre of $I \times I$ and consider the open sets

$$\begin{aligned}\tilde{A} &= \{x \in I \times I \colon \|x - p\| > 1/3\}\\ \tilde{B} &= \{x \in I \times I \colon \|x - p\| < 2/3\}.\end{aligned}$$

Let $q\colon I \times I \to \mathbb{T}^2$ be the quotient map and $A = q(\tilde{A})$, $B = q(\tilde{B})$. Thus $A \cap B$ is an annulus and $\mathbb{T}^2 = A \cup B$. We would like to derive the fundamental group of $\mathbb{T}^2$ using the Seifert-van Kampen theorem (recall that we already know this fundamental group, this is only to get a more insightful description). For this, choose a basepoint $x_0 \in A \cap B$. See Figure 25.29 for an illustration.

As usual, we denote by ι_A, ι_B the inclusions of A and B in $\mathbb{T}^2$, and by ι_{AB}, ι_{BA} the inclusions of $A \cap B$ in A and B, respectively.

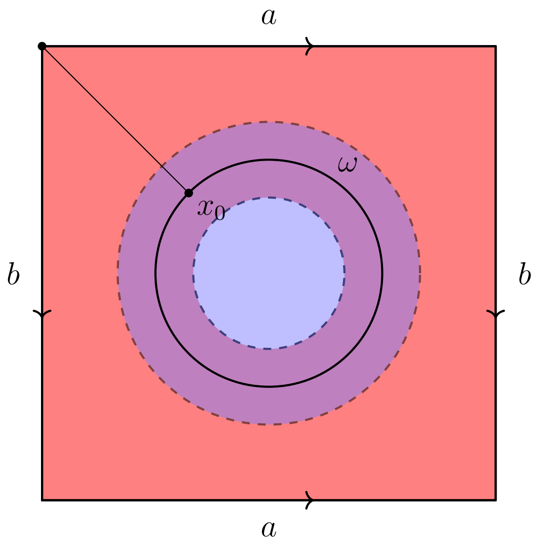


Figure 25.29: An open cover of the torus.

$$\begin{tikzcd}
 & \pi_1(A, x_0) \arrow[dr] \arrow[drr, bend left, "(\iota_A)_*"] & & \\
\pi_1(A \cap B, x_0) \arrow[ur, "(\iota_{AB})_*"] \arrow[dr, "(\iota_{BA})_*"'] & & \pi_1(A, x_0) * \pi_1(B, x_0) \arrow[r, dashed] & \pi_1(\mathbb{T}^2, x_0) \\
 & \pi_1(B, x_0) \arrow[ur] \arrow[urr, bend right, "(\iota_B)_*"'] & &
\end{tikzcd}$$

The set $A \cap B$ is an annulus, that retracts to a circle. The fundamental group is generated by a loop ω at x_0. Since B is a disk, it is contractible, $\pi_1(B, x_0) = \mathbf{1}$ and

$$\pi_1(A, x_0) * \pi_1(B, x_0) = \pi_1(A, x_0).$$

Moreover,

$$(\iota_{BA})_*([\omega]) = [\iota_{BA} \circ \omega] = e_{\pi_1(B, x_0)},$$

so that the normal subgroup N of $\pi_1(A, x_0)$ is generated by

$$\iota_{AB}([\omega])_* \iota_{BA}([\omega])_*^{-1} = (\iota_{AB})_*([\omega]).$$

This is where things become interesting: what is $\pi_1(A, x_0)$, and how is $(\iota_{AB})_*([\omega])$ represented in this group? Notice that A is homotopy equivalent to a torus with a missing point in the middle (use the straight-line homotopy), and by a previous exercise this deformation retracts onto the figure-eight $S^1 \vee S^1$. Moreover, the fundamental group of this figure-eight is the free group generated by the loops a and b, that is, it consists of words in the letters $[a]$ and $[b]$. To get an explicit representation with respect to the basepoint x_0, choose a path h from x_0 to the intersection y_0 of a and b and define the loops $\gamma_a = h * a * \overline{h}$ and $\gamma_b = h * b * \overline{h}$. We then have the basepoint-change isomorphism

$$\beta_h \colon \pi_1(A, x_0) \to \pi_1(A, y_0),$$

that maps $[\gamma_a]$ to $[a]$ and $[\gamma_b]$ to $[b]$, as shown in a previous lecture. Inside A, the loop ω can now be *factored* as follows:

$$\omega \overset{\partial}{\simeq} \gamma_a * \gamma_b * \gamma_a^{-1} * \gamma_b^{-1},$$

which leads to a representation

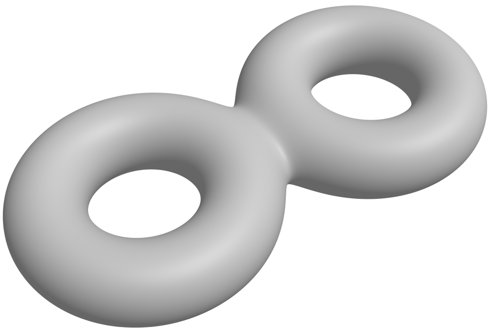
$$(\iota_{AB})_*([\omega]) = [\iota_{AB} \circ \omega] = [\gamma_a] \bullet [\gamma_b] \bullet [\gamma_a]^{-1} \bullet [\gamma_b]^{-1}.$$

If, by abuse of notation, we denote $a = [\gamma_a]$ and $b = [\gamma_b]$, then we can say that the fundamental group of $\mathbb{T}^2$ is *presented* as

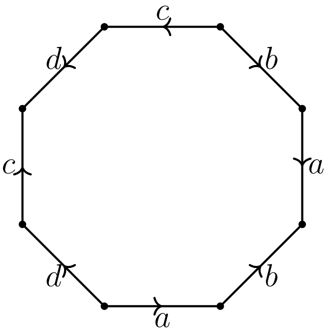
$$\pi_1(\mathbb{T}^2) = \langle a \rangle * \langle b \rangle / \langle\langle aba^{-1}b^{-1} \rangle\rangle.$$

The elements a, b are the **generators** and $aba^{-1}b^{-1}$ is a **relation**. Setting $aba^{-1}b^{-1} = 1$ amounts to requiring $ab = ba$, so that imposing this relation makes the group abelian. The resulting group is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

The torus is just a special case of a whole class of surfaces. Consider the surface S_g with g "handles". For example, the double torus:



This surface can be represented by identifying sides on an octagon:



Identifying the edges as indicated shows that the boundary is homotopic to a wedge of four spheres, $S^1 \vee S^1 \vee S^1 \vee S^1$, and in particular that all the corner points are identified with the same point. More generally, given a polygon with $4g$ edges, identifying the edges gives a boundary that is a wedge of $2g$ circles, and the resulting surface is called M_g, with g the **genus** of the surface. Using exactly the same proof as with the torus, we arrive at a fundamental group that is given by generators a_i, b_i for $1 \leq i \leq g$, and relations by the products of the elements $[a_i, b_i] := a_i b_i a_i^{-1} b_i^{-1}$, the *commutators*. A group with this structure is said to have the *presentation*

$$\langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

In general, a group G has a **presentation**

$$\langle S \mid R \rangle,$$

where S is a set of **generators** and R is a set of **relators**, is G is the free group generated by the elements of S modulo the normal subgroup generated by R,

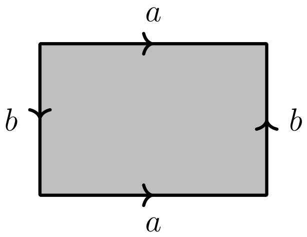
$$G = \langle S \rangle / \langle\langle R \rangle\rangle.$$

Example 25.1. The group $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ has the presentation $\langle a \mid a^2 \rangle$.

A group is called **finitely generated** if it has a finite set S of generators, and **finitely presented**, if both S and R are finite sets.

Example 25.2. Just as there are different ways of describing a topological space as a CW complex, there are different ways to "present" the fundamental group. We discuss this using an illustrative example, the **Klein bottle** K.

The image shows an attempted embedding of the Klein bottle into $\mathbb{R}^3$; this is not possible without self-intersections. As a CW complex, the Klein bottle is usually described like a Möbius strip, but with the top and bottom sides identified as well.

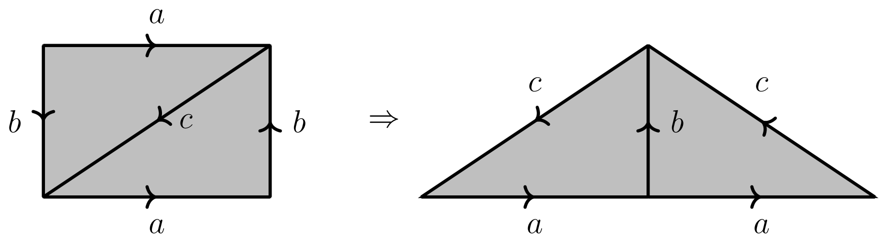


The underlying 1-skeleton X^1 consists of two loops, a and b, while there is only one vertex (by following the identification of the boundaries of the rectangle as indicated by the arrows, one sees that all the corners are collapsed to a single point). Therefore, the generators are the classes corresponding to the cycles a and b (which we will also denote by a and b). The single relation is the loop that forms the boundary of the rectangle and is given by $baba^{-1}$ (formally, the class in the fundamental group of X^1 that is generated by the loop $b * a * b * \overline{a}$). We therefore get a presentation

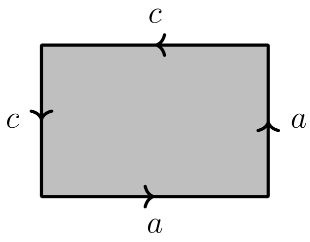
$$\langle a, b \mid baba^{-1} \rangle.$$

In other words, all the elements in this group are words in a and b (or equivalently, binary sequences), where every occurrence of $baba^{-1}$ is replaced with the empty word.

One might, of course, ask whether this group looks like a more familiar group, or whether it can be described in a simpler way. One way to arrive at such a simpler representation is to use a different CW-complex representation.



In this case, we can add an additional cycle c and then remove the cycle b. The resulting picture can then be visualized as follows.



The resulting group presentation is then

$$\langle a, c \mid a^2c^2 \rangle.$$

This is easier to interpret. Of course, simply setting $c := b^{-1}a^{-1} = a^{-1}b$ we see that we can represent every word in a and b as a word in a and c, and that $c^2 = a^{-2}$, so we can just get the alternative presentation on a purely group-theoretic level. Each such presentation corresponds to a different way of describing a topological space.

Lecture 26

Consider a graph $X = X^1$ consisting of a set of vertices $V = X^0$ and edges $(D^1_\alpha, \varphi_\alpha)$, where $\varphi_\alpha \colon S^0_\alpha \to X^0$ is the attaching map that assigns to each interval D^1_α its endpoints in the graph. Recall the characteristic map $\Phi_\alpha \colon D^1_\alpha \to X^1$ that maps each 1-cell to its image in the graph. We use the term *edge* for both a pair $(D^1_\alpha, \varphi_\alpha)$, which records *combinatorial* information (e.g., which are the endpoints), and for the image $\Phi_\alpha(D^1_\alpha)$ as a topological subspace of the graph.

In the following we use the convention that an *edge-path* in a graph is a path that can be written as a concatenation of edges:

$$\gamma = e_1 * \cdots * e_m,$$

where each e_i is an edge (in the subspace-sense). Similarly, and edge loop is an edge-path that ends where it starts.

26.1 From CW complexes to groups and back

Given a CW complex X and $x_0 \in X$, we can compute a presentation of the fundamental group $\pi_1(X, x_0)$. As the path-components that do not contain x_0 do not enter the fundamental group, we may replace X with the path component containing x_0. In addition, we can move the basepoint to lie in X^1 (or even X^0), as this does not change the structure of the fundamental group. Finally, we can restrict to the 2-skeleton, and hence assume without lack of generality that $X = X^2$ is a path-connected, two-dimensional CW complex. To compute the fundamental group we proceed as follows:

1. Find a spanning tree of $T \subset X^1$. This can be done, for example, using Dijkstra's algorithm. Let $\mathcal{A}$ be the set (not union!) of edges that are *not* in the tree. Pasting such an edge to the graph T gives a subgraph that is homotopic to a circle S^1, i.e., an edge-cycle. As shown in the exercises, we can describe the fundamental group of X^1 as generated by these edge-cycles.

   $$\pi_1(X^1, x_0) \cong *_{e \in \mathcal{A}} \mathbb{Z}.$$

   Every edge not in T gives a loop when adding it to T, and conversely every loop in X^1 based at x_0 is homotopic to a combination of such edge-cycles (loops that consist of traversing a cycle that arises by adding $a \in \mathcal{A}$ along edges).

2. Let $e^2_\alpha \subset X^2$ (here we identify the open 2-cells with their images in X^2) be a 2-cell and

$$\varphi_\alpha \colon S^1_\alpha \to X^1$$

the attaching map. Recall that $\gamma_\alpha(t) = \varphi_\alpha(\exp(2\pi it))$ is a loop, and hence homotopic to an edge-loop (a loop consisting of edges). Let $x_1 \in \varphi_\alpha(S^1_\alpha)$ and let $g_\alpha \colon I \to X^1$ be a path with $g_\alpha(0) = x_0$ and $g_\alpha(1) = x_1$. Then

$$\omega_\alpha = [g_\alpha * \gamma_\alpha * \overline{g}_\alpha] \in \pi_1(X^1, x_0)$$

and therefore corresponds to a reduced word u_α in $\mathcal{A}$. Set $U = \{u_\alpha\}_\alpha$.

We claim that

$$\pi_1(X, x_0) \cong \pi_1(X^1, x_0)/\langle\langle U\rangle\rangle,$$

or in other words, that the fundamental group of X with base x_0 is presented as $\langle \mathcal{A} | U\rangle$. In fact,

- The union of the cells e^2_α together with the paths joining them to x_0 form a contractible subcomplex: $\pi_1(A, x_0) \cong \mathbf{1}$.

- Choose points $y_\alpha \in e^2_\alpha$ inside each of the cells e^2_α and define the subset $B = X^2 - \bigcup_\alpha \{y_\alpha\}$. Then B retracts to X^1 (we poke a "hole" into each of the 2-cells attached to X^1), and $\pi_1(B, x_0) \cong \pi_1(X^1, x_0)$.

- We have $X^2 = A \cup B$ and $A \cap B$ consists of precisely those edge-cycles starting at x_0 that make up loops homotopic to the boundaries of 2-cells, or in other words, the images of S^1_α under the attaching maps. Therefore, each element of $\pi_1(A \cap B, x_0)$ represents a word in U.

- The fundamental group of X is therefore given as

$$\pi_1(X, x_0) \cong \pi_1(X^2, x_0) \cong \pi_1(A, x_0) * \pi_1(B, x_0)/\langle\langle U\rangle\rangle \cong \pi_1(X^1, x_0)/\langle\langle U\rangle\rangle.$$

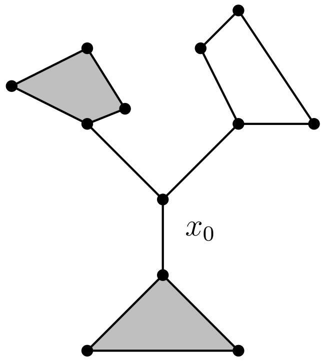


Figure 26.30: The graph X^2

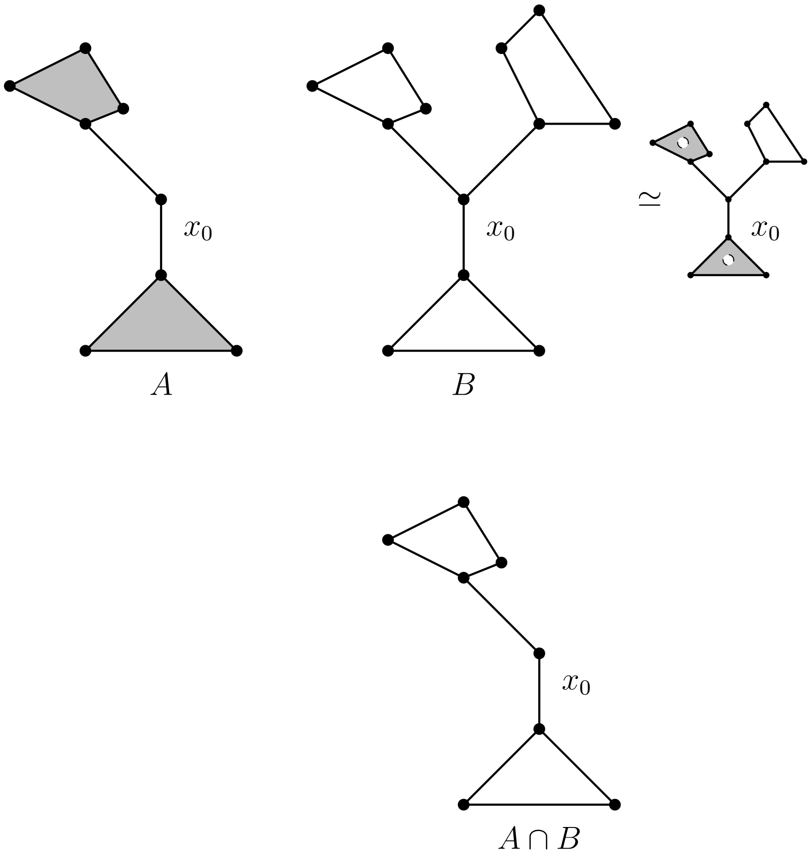


Figure 26.31: The subcomplexes A, B and $A \cap B$

The construction is best visualized as in Figures 26.30 and 26.31. In summary:

- Every **cycle** in the underlying graph X^1 corresponds to a loop based at x_0 that moves along edges from x_0 to the cycle, around the cycle, and back to x_0. Every such cycle corresponds to a **generator** of the fundamental group $\pi_1(X^2, x_0)$;
- Every **loop** in X^2 can be represented as a combination of such cycles-paths along edges. This corresponds to a reduced **word** in the generators of $\pi_1(X^2, x_0)$;
- A loop is **null-homotopic** if it is homotopic to the boundary of a 2-cell in X^2. Such loops corresponds to a **relation** on the set of words in $\pi_1(X^2, x_0)$.

The Seifert-van Kampen Theorem merely provides a means to formalizing the above intuitive procedure.

Example 26.1. Recall the characterization of real projective space as CW complex. Recall the cell decomposition of $\mathbb{RP}^2$ into one 0-cell, one 1-cell and one 2-cell, which can be visualized as follows.

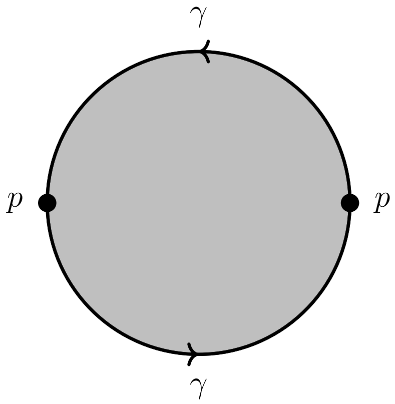
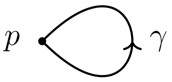


Figure 26.32: Cell decomposition of $\mathbb{RP}^2$.

Even though we see two points and two arcs labelled with γ, the points are identified to make one point, and the lines are identified (glued together) along the direction of the arrow. The 1-skeleton X^1 of this is just a loop consisting of a single edge, and a spanning tree consists of the only vertex in this graph. The generator of the fundamental group is thus this one cycle,



whose class we denote by a (say). For the relation, we look at the loop that bounds the 2-cell: as seen in the image, this loop consists of going around the cycle twice, so it is represented by a^2. Therefore, the fundamental group is presented by $\langle a \mid a^2 \rangle$, and the corresponding group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

By now we should have an idea of how to get a group out of a CW complex. Conversely, any group presentation leads to a topological space (in fact, a surface) whose fundamental group is isomorphic to the given group.

Theorem 26.2. *For every group G there exists a path-connected two-dimensional CW complex X_G such that*

$$\pi_1(X_G) \cong G.$$

Proof. Consider a presentation of the group (generators and relators). Construct the one skeleton X^1 of X_G as a wedge (one point union) of circles S^1, with one circle per generator. Every relator describes a loop in X^1: for example if $ab^{-1}c^2$ is a relator, then the loop is given by going around a once, around b once in the opposite direction, and then twice around c. For each such relator take a 2-cell D^2_α with boundary S^1_α and define an attaching map

$$\varphi_\alpha \colon S^1_\alpha \to X^1$$

that maps the circle onto the loop specified by the relators. The resulting CW-complex $X = X^2$ is then a two-dimensional CW complex whose fundamental group is, by construction, isomorphic to G. $\square$

Example 26.3. Consider the group $G = \langle a \mid a^n \rangle$ for some integer n. This is isomorphic to the additive group $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n. To construct a corresponding topological spaces, we begin with one copy of a circle, $X^1 = S^1$, and set $a = [\omega]$, where $\omega\colon I \to S^1$ is the map $\omega(t) = \exp(2\pi i t)$. We then attach a 2-cell D^2 via the map $\varphi\colon S^1 \to X^1$ in such a way that the loop $\gamma(t) = \varphi(\exp(2\pi i t))$ satisfies $[\gamma] = a^n$. This is achieved by setting

$$\varphi(z) = z^n,$$

which is precisely the usual n-fold covering map $S^1 \to S^1$. The case $n = 2$ gives rise to $\mathbb{RP}^2$, but for $n > 2$ we cannot embed the resulting space in $\mathbb{R}^3$.

Lecture 27

27.1 Orientable and non-orientable surfaces

In what follows, we recall that an n-dimensional (topological) manifold M is a topological Hausdorff space that is second countable (that is, with a countable base), and such that each point has a neighbourhood homeomorphic to some $\mathbb{R}^n$. A **surface** is a 2-dimensional manifold.

27.2 Classification of surfaces

Lecture 28

28.1 The Universal Covering