2024-10-01 LECTURE 2 SAUL SCHLEIMER MA3F1

Exercise: $S^2 \not\cong \mathbb{T}^2 \cong S^1 \times S^1$  $\not\cong$ .

IN FACT: SUPPOSE X IS A TOP. SPACE. THEN $S^2 \not\cong X \times X$.

EXERCISE: (i) $X \times Y \cong Y \times X$ (Also, $X \cong X$.)

(ii) $(X \times Y) \times Z \cong X \times (Y \times Z)$ for SPACES X, Y, Z.

1 NEW SPACES for OLD

DEF: SUPPOSE X, Y ARE SPACES.

$X \sqcup Y$ IS THE DISJOINT UNION of X AND Y.

THIS HAS UNDERLYING SET $X \times \{0\} \cup Y \times \{1\}$ AND

OPEN SETS $U \times \{0\}$ AND $V \times \{1\}$ FOR ANY $U \subset X$, $V \subset Y$ OPEN.

EXAMPLE: $S^0 = \{pt\} \sqcup \{pt\}$. $\bullet \quad \bullet$ $\cong \{-1, +1\}$

DEF: SUPPOSE X IS A SPACE. SUPPOSE $E \subset X \times X$ IS AN EQUIV RELATION. ~~DEFINE~~

$$X/E = \{ [x]_E \mid x \in X \}$$ TO BE THE SET OF EQUIV. CLASSES

DEFINE $q_E : X \longrightarrow X/E$ by $q_E(x) = [x]_E$.

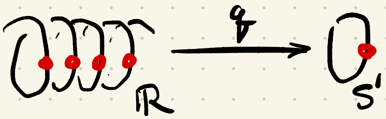
DEFINE $V \subset X/E$ TO BE OPEN IN THE QUOTIENT TOPOLOGY iff $q_E^{-1}(V)$ IS OPEN IN X.

NOTE
$$q_E^{-1}(V) = \{ x \in X \mid q_E(x) \in V \}$$
$$= \{ x \in X \mid [x]_E \in V \}$$
$$= \bigcup_{[x]_E \in V} [x]_E$$

EXERCISE: $q_E : X \longrightarrow X/E$ IS CONTINUOUS.

EXAMPLE: DEFINE AN EQU. REL ON $\mathbb{R}$ by xEy iff $y-x\in\mathbb{Z}$.
THEN $\mathbb{R}/E \cong S^1$

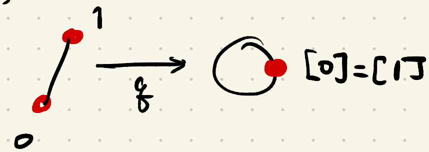
PICTURE



EXAMPLE: DEFINE AN EQU. REL on $I=[0,1]$ by

$$xEy \text{ iff } (x=y \text{ OR } \{x,y\}=\{0,1\}).$$

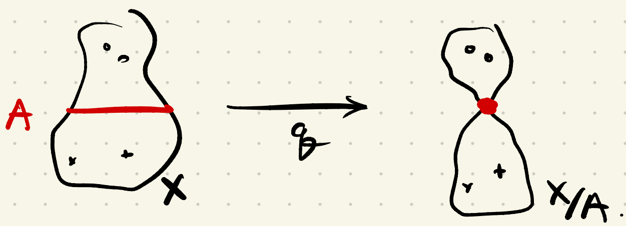
THEN $I/E \cong S^1$ **PICTURE**:



THIS IS "GLUING"

NOTATION: SUPPOSE $A\subset X$ IS SUBSET. DEFINE THE QUOTIENT X/A VIA THE RELATION xE_Ay iff $(x=y$ OR $x,y\in A)$.

PICTURE



THIS IS "CRUSHING"

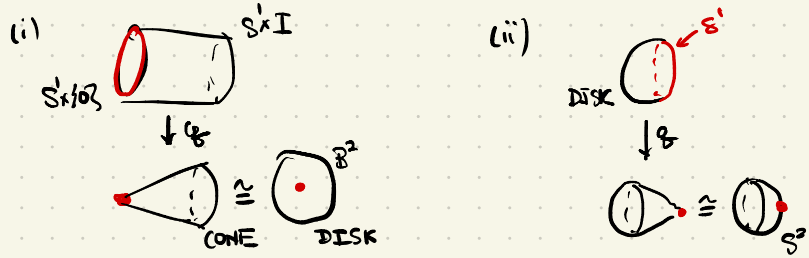
IMPORTANT EXERCISE

NOTE $S^{n-1}\subset B^n$. PROVE THE FOLLOWING

(i) $S^{n-1}\times I \big/ {S^{n-1}\times\{0\}} \cong B^n$

(ii) $B^n/S^{n-1} \cong S^n$.

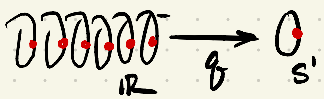
PICTURES:



NOTATION: SUPPOSE X IS A SPACE. SUPPOSE $G \curvearrowright X$ IS A GROUP ACTING ON X. DEFINE $x E_G y$ iff $y = g \cdot x$ for SOME $g \in G$. DEFINE $X/G = X/E_G$

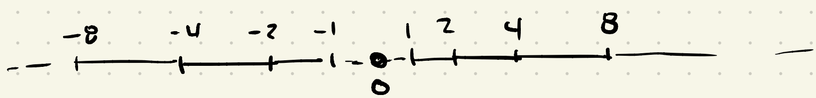
EXAMPLE: SUPPOSE $\mathbb{Z}$ ACTS ON $\mathbb{R}$ BY TRANSLATION. THAT IS $n \cdot r = r + n$.

EXERCISE: $\mathbb{R}/\mathbb{Z} \cong S^1$.



CAREFUL: THE NOTATION IS AMBIGIOUS! Suppose $\mathbb{Z}$ ACTS ON $\mathbb{R}$ BY SCALING THAT IS $n \cdot r = 2^n \cdot r$.

EXERCISE: NOW $\mathbb{R}/\mathbb{Z}$ IS NOT HAUSDORFF!



BUT $(\mathbb{R} - \{0\})/\mathbb{Z} \cong S^1 \sqcup S^1$ IS HAUSDORFF.

FINAL EXAMPLE: HAVE $\mathbb{Z}^n$ ACT ON $\mathbb{R}^n$ BY TRANSLATION: $v \cdot x = x + v$.

THEN $\mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n$.

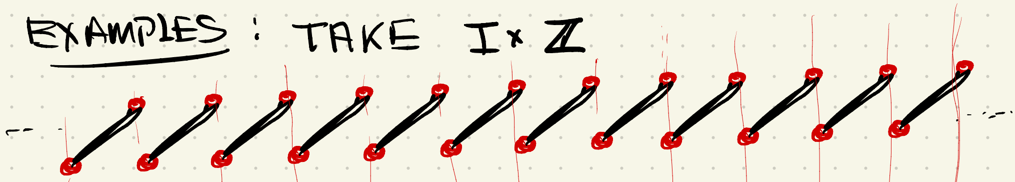
② CUT-AND-PASTE: THE OPERATIONS of

(i) DISJOINT UNION and $\emptyset$

(ii) QUOTIENTS (GLUING)

LETS US BUILD MANY SPACES:

EXAMPLES: TAKE $I \times \mathbb{Z}$



AND GLUE ENDPOINTS: $\downarrow q$

EXERCISES: BUILD $\mathbb{R}^n$ OUT of n-CUBES (COPIES of $I^n = I^{n-1} \times I$)

BUILD $\mathbb{R}^2$ OUT of TRIANGLES, HEXAGONS, -- PENTAGONS..

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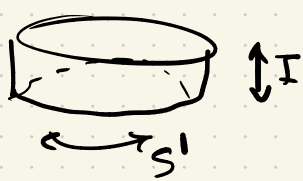
③ OUR FRIEND THE SQUARE

GOT HERE

WE CAN ALSO GLUE PARTS of A SPACE TO EACH OTHER:

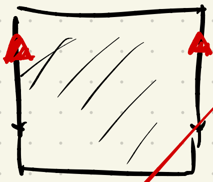
DEFINE: $A^2 = S^1 \times I$ PICTURE



DEFINE $X = I^2$, AND $E \subset X \times X$ BY

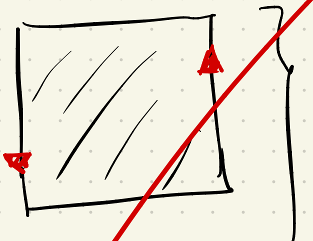
$u E v$ iff $(u = v$ OR $(\{u_1, v_1\} = \{0,1\}$ AND $u_2 = v_2))$

PICTURE:



$I^2/E \cong A^2$

ANOTHER EXAMPLE



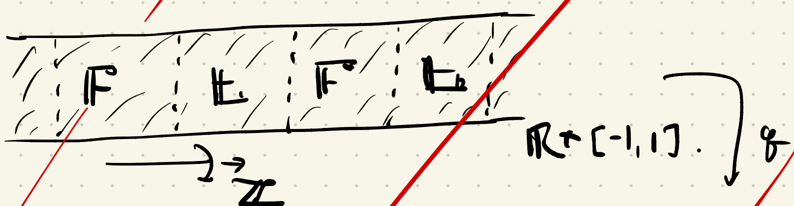
$u E v$ iff $\Big(u=v \text{ OR } \big(\{u_1,v_1\}=\{0,1\} \text{ AND } u_2+v_2=1 \big)\Big)$

THIS IS THE MOBIUS BAND

ANOTHER DEF: $X=\mathbb{R}\times[-1,1]$, $\mathbb{Z}$ ACTS BY

$$n\cdot(x,y) = (x+n, (-1)^n\cdot y).$$

PICTURE



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