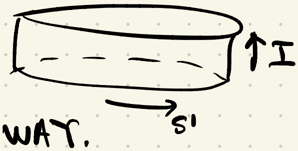
2024-10-03 LECTURE 3 MA3F1 SAUL SCHLEIMER

(1) OUR FRIEND THE SQUARE

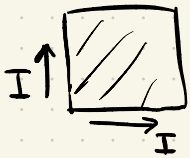
WE CAN ALSO GLUE PARTS of A SPACE TOGETHER.

DEF: $A^2 = S^1 \times I$ THE ANNULUS

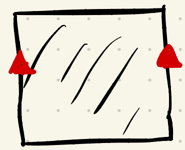


WE CAN BUILD A^2 ANOTHER WAY.

SET $X = I^2$



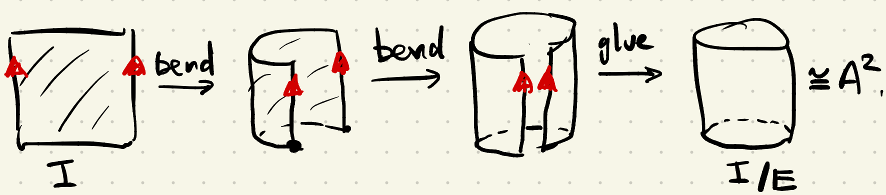
THE SQUARE. WE GLUE THE LEFT and RIGHT SIDES AS FOLLOWS



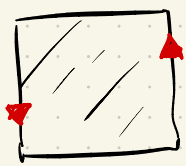
FORMALLY: DEFINE $E \subset X \times X$ BY

$$xEy \text{ iff } \left(x = y \text{ OR } \left(\{x_1, y_1\} = \{0,1\} \text{ and } x_2 = y_2\right)\right)$$

PICTURES



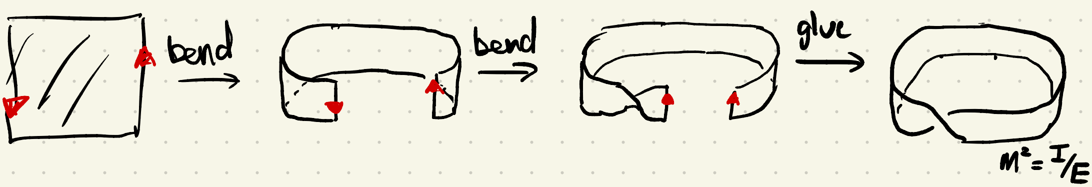
ANOTHER EXAMPLE:



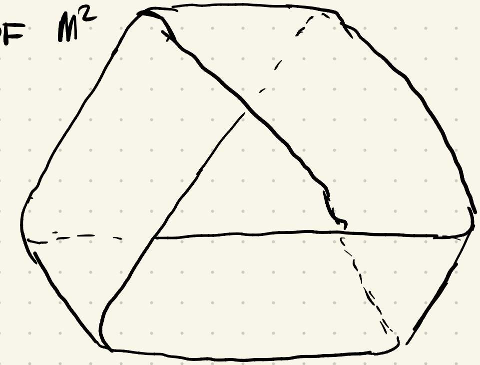
DEFINE xEy by

$$\left(x = y \text{ OR } \left(\begin{matrix}\{x_1, y_1\} = \{0,1\} \text{ AND} \\ x_2 + y_2 = 1\end{matrix}\right)\right)$$

Now $I/E = M^2$ IS THE MOBIUS BAND

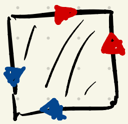


YET ANOTHER PICTURE OF M^2

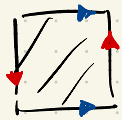


EXERCISE: FIND ALL "EDGEWISE" QUOTIENTS of THE SQUARE. GLUING EDGES IN PAIRS.

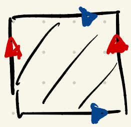
(1) S^2



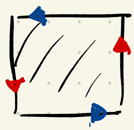
(3) K^2



(2) T^2



(4) P^2



UP TO HOMEOMORPHISM.

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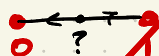
(2) RETRACTS: SUPPOSE X IS A SPACE AND $A \subset X$ IS A SUBSET. SAY A IS A RETRACT of X IF THERE IS A MAP $f: X \to A$ SO THAT $f|A = Id_A$.
[THAT IS: $f(a) = a$ for ALL $a \in A$]

EXAMPLE: $\mathbb{R}^2 - \{0\}$ RETRACTS TO S^1 via
$f: \mathbb{R}^2 - \{0\} \longrightarrow S^1$ BY $f(x) = x/|x|$.

INTERMEDIATE VALUE THEOREM: $X = [0,1]$ DOES NOT RETRACT TO $A = \{0,1\}$.



[THIS IS A NON-EXISTANCE THEOREM]

NO RETRACT THM [BROUWER] B^n DOES NOT RETRACT TO $S^{n-1} \subset B^n$.

THIS (PLUS WORK) PROVES INVARIANCE of DOMAIN!

IN THIS MODULE WE'LL DEAL WITH THE CASE of $n=2$.

PICTURE

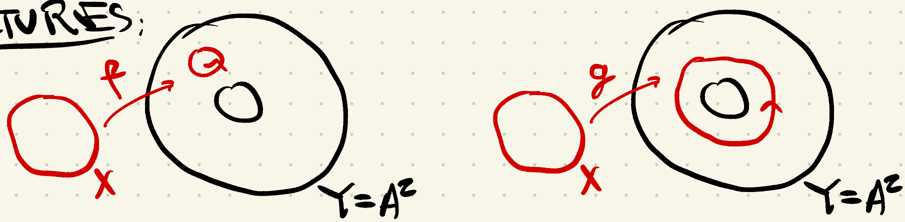
REMARK: WE'LL RETURN TO RETRACTS AND DEFORMATION RETRACTS.

(3) **HOMOTOPIES**: SUPPOSE X, Y ARE SPACES. A MAP $F: X \times I \longrightarrow Y$ IS CALLED A **HOMOTOPY**. DEFINE $f_t: X \longrightarrow Y$ BY $f_t(x) = F(x,t)$. CALL F A HOMOTOPY **FROM** f_0 **TO** f_1.

EXAMPLE: $X = S^1$, $Y = \mathbb{R}^2$

DEF: SUPPOSE $f, g: X \to Y$ ARE MAPS. CALL f, g **HOMOTOPIC** IF THERE IS SOME HOMOTOPY F WITH $f = f_0$ AND $g = f_1$. IF SO WRITE $f \simeq g$

PICTURES:



$f \not\simeq g$.

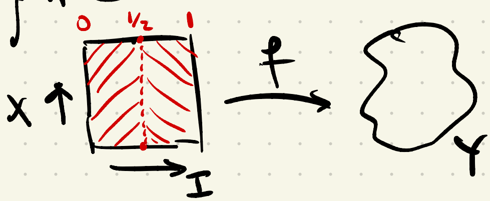
EXERCISE: $f \simeq g$ IS AN EQU. RELATION ON $\mathrm{MAP}(X,Y)$

WILL NEED THE

GLUING LEMMA: SUPPOSE X, Y ARE SPACES. SUPPOSE $F: X \times I \longrightarrow Y$ IS A FUNCTION SO THAT

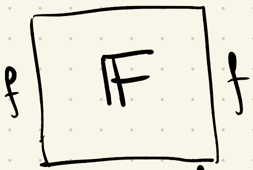
(i) $F \mid X \times [0, \tfrac{1}{2}]$
(ii) $F \mid X \times [\tfrac{1}{2}, 1]$

ARE CONTINUOUS. THEN F IS CONTINUOUS



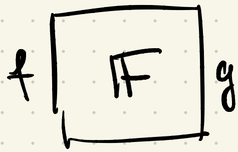
PICTURES

REFLEXIVE

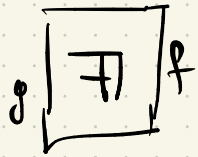


$F(x,t) = f(x)$

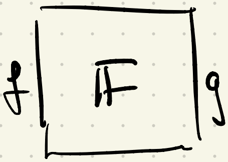
SYMMETRIC



GIVES



TRANSITIVE



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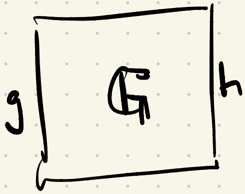

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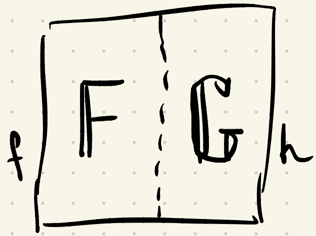

NOTE: HOMEOMORPHISMS DO NOT "CRUSH" OR "TEAR". CONTINUOUS MAPS CAN "CRUSH" BUT STILL CANNOT "TEAR"

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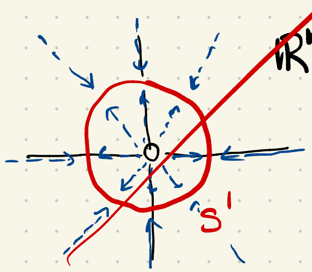
(4) DEFORMATION RETRACT: SUPPOSE X IS A SPACE, $A \subset X$ A SUBSET. SUPPOSE $f: X \to A$ IS A RETRACT. WE CALL f A DEF. RETRACT IF THERE IS A HOMOTOPY $F: X \times I \longrightarrow X$ SO THAT $f_0 = Id_X$, $f_1 = f$ AND $f_t | A = Id_A$ FOR ALL $t \in I$.

EXAMPLES: $\mathbb{R}^n$ DEF RETRACTS TO $\{0\}$

$\mathbb{R}^n - \{0\}$ " " S^{n-1}



PICTURE FOR $n=2$.

MORALLY: F MOVES $x \in X$ ALONG A PATH TO $f(x) \in A$.

MORE EXAMPLES: B^n DOES NOT DEF RET. TO S^{n-1}

M^2, A^2 DO DEF RETRACT TO COPIES of S^1