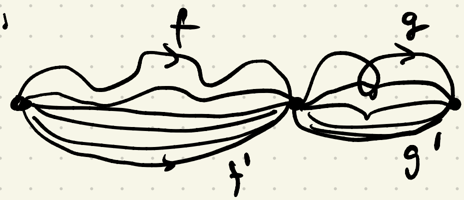
2024-10-08 LECTURE 5 MA3F1 SAUL SCHLEIMER

(1) **PATHS**: FOR $f, g : I \to X$ WITH $f(1) = g(0)$
DEFINE $f * g : I \to X$ BY $(f*g)(t) = \begin{cases} f(2t) & t \leq 1/2 \\ g(2t-1) & t \geq 1/2 \end{cases}$

EXERCISE: IF $f \simeq_\partial f'$, $g \simeq_\partial g'$ AND $f(1) = g(0)$
THEN $f'(1) = g'(0)$ AND $f * g \simeq_\partial f' * g'$

PICTURE:



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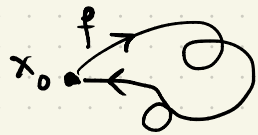
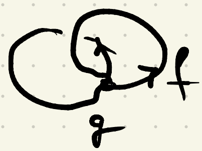
(2) **LOOPS**: SUPPOSE X IS A SPACE, $x_0 \in X$ IS A BASEPOINT. CALL THE PAIR (X, x_0) A **POINTED** SPACE.

DEF A PATH $f : I \to X$ IS A **LOOP BASED AT** x_0 IF $f(0) = f(1) = x_0$. **PICTURE**



[DEF: LOOPS(X, x_0) BELOW]

NOTE: IF f, g ARE LOOPS AT x_0 THEN SO IS $f * g$. **PICTURE**:



IN $\mathbb{R}^3$ SAY.

NOW DEFINE
$$[f] = \{ f' \text{ A LOOP AT } x_0 \mid f' \simeq_\partial f \}$$
THE EQUIV. CLASS OF f.

PICTURE

CAN WIGGLE f TO f' BUT CANNOT "JUMP OVER HOLES".

③ THE FUNDAMENTAL GROUP

SUPPOSE (X, x_0) IS A POINTED SPACE.

DEFINE

$$\pi_1(X, x_0) = \left\{ [f] \;\middle|\; \begin{array}{l} f : I \longrightarrow X \\ \text{IS LOOP BASED} \\ \text{AT } x_0 \end{array} \right\}.$$

DEFINE $[f] \cdot [g] = [f * g]$

EXERCISE: THIS IS WELL-DEFINED.

PROP 1.3: $(\pi_1(X, x_0), \cdot)$ IS A GROUP.

EXAMPLE: $\pi_1(\mathbb{R}^n, 0)$ IS THE TRIVIAL GROUP.

[NO HOLES!].

PROOF of 1.3: SUPPOSE f, g, h ARE ANY LOOPS AT x_0

SUPPOSE e IS THE CONSTANT LOOP AT x_0.

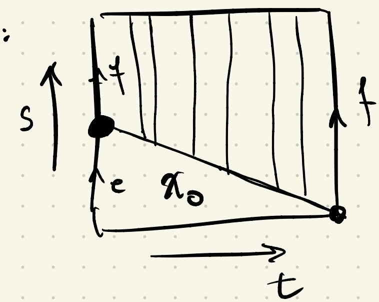
(i) CLOSURE: $f * g$ IS A LOOP AT x_0.

THUS $[f] \cdot [g] \in \pi_1(X, x_0)$.

(ii) IDENTITY:

CLAIM: $[f] \cdot [e] = [e] \cdot [f] = [f]$.

PROOF:



DEFINE

$$F(s,t) = \begin{cases} x_0, & \text{if } 2s + t \leq 1 \\ f\left(\frac{2s+t-1}{t+1}\right), & \text{if } 2s + t \geq 1 \end{cases}$$

[FOR EXAMPLE

THUS $e * f \simeq_\partial f$. SIMILARLY, $f * e \simeq_\partial f$.

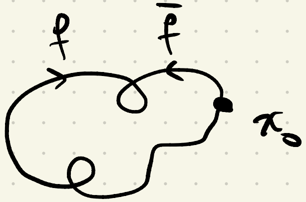
SO $[e]\cdot[f] = [f][e] = [f]$. //

(ii) <u>INVERSES</u>. DEFINE $\bar{f}: I \to X$ BY

$\bar{f}(t) = f(1-t)$.

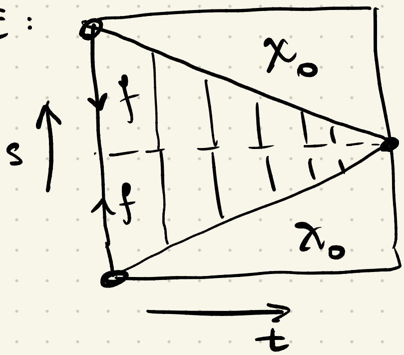
$[\text{NOTE } \bar{\bar{f}} = f]$.

<u>PICTURE</u>:



<u>CLAIM</u>: $[f]\cdot[\bar{f}] = [\bar{f}][f] = [e]$.

<u>PROOF</u>:



IN COORDINATES

$$F(s,t) = \begin{cases} x_0, & \text{if } 2s \le t \\ x_0, & \text{if } 2s \ge 2-t \\ f(2s-t), & \text{if } t \le 2s \le 1 \\ f(2+t-2s), & \text{if } 1 \le 2s \le 2-t \end{cases}$$

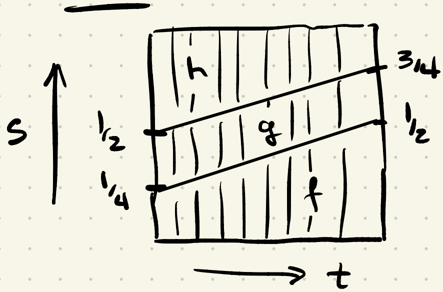
THUS $f * \bar{f} \simeq_\partial e$. SIMILARLY $\bar{f} * f \simeq_\partial e$.

So $[f][\bar{f}] = [\bar{f}][f] = [e]$ //

(iii) <u>ASSOCIATIVITY</u>:

<u>CLAIM</u>: $([f][g])[h] = [f]([g][h])$.

<u>PROOF</u>: WE PROVE $(f*g)*h \simeq_\partial f*(g*h)$ VIA



THE PARAMETERISATION IS A BIT WORSE THAN THE CASE OF THE IDENTITY (ABOVE). EXERCISE! //

DEF: SUPPOSE (X, x_0) IS PATH-CONNECTED. WE CALL (X, x_0) SIMPLY-CONNECTED IF $\pi_1(X, x_0) \cong \mathbb{1}$ [TRIVIAL GROUP]

IMPORTANT EXAMPLE

$X = S^1$, $x_0 = (1,0)$.

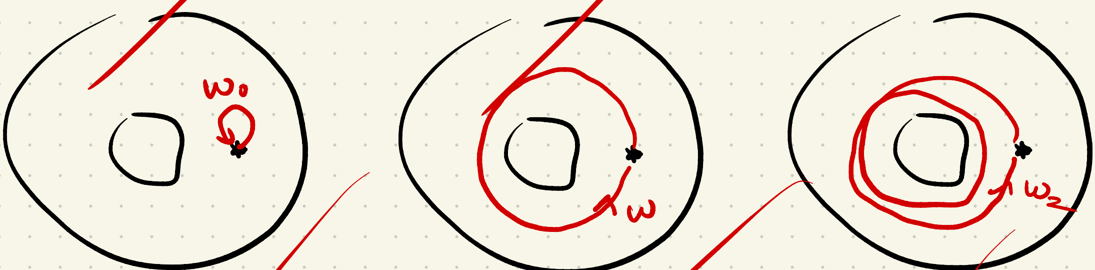
DEFINE $\omega_n(t) = (\cos(2\pi n t), \sin(2\pi n t))$.

NOTE $\omega_n : I \longrightarrow S^1$ IS A LOOP.

DEF $e = \omega_0$, $\omega = \omega_1$

THEOREM 1.7: $\pi_1(S^1, x_0)$ IS ISOMORPHIC TO $\mathbb{Z}$ GEN. BY $[\omega]$.

PICTURES: WE THICKEN S^1 TO GET $A^2 = S^1 \times I \simeq S^1$



SO ω_n WINDS n TIMES AROUND THE ANNULUS.

EXERCISES: $\omega_1 * \omega_{-1} \underset{\partial}{\simeq} \omega_0$

$$\bar{\omega}_1 = \omega_1$$

$$\omega_1 * \omega_1 \overset{(*)}{=} \omega_2 \quad (*) \text{ SPECIAL CASE!}$$

$$\omega_n * \omega_m \underset{\partial}{\simeq} \omega_{n+m}$$

INTRODUCED NOTATION:

$$\text{LOOPS}(X, x_0) = \left\{ f : I \to X \;\middle|\; \begin{array}{l} f(0) = f(1) \\ = x_0 \end{array} \right\}$$

THE SET of LOOPS.

SO $*: \text{LOOPS}(X,x_0) \times \text{LOOPS}(X,x_0) \longrightarrow \text{LOOPS}(X,x_0)$

IS BINARY OP. BUT (*) NO IDENTITY

(*) NO INVERSES

(*) NO ASSOCIATIVITY!

BUT <u>HAS</u> THESE THINGS "UP TO HOMOTOPY" SO MOD OUT

$$\left.\begin{array}{ccc} \text{LOOPS}(X,x_0) & \longrightarrow & \pi_1(X,x_0) \\ f & \longmapsto & [f]. \end{array}\right\} \text{ AND WIN}$$