2024-10-15 LECTURE 8 MA3F1 SAUL SCHLEIMER

35 STUDENTS

① PUSHING DOWN

SUPPOSE $p: \tilde{X} \to X$ IS A COVERING MAP.

LEMMA Ⓐ: SUPPOSE $\hat{f}, \hat{g}: Y \to \tilde{X}$ ARE MAPS. IF $\hat{f} \simeq \hat{g}$ THEN $p \circ \hat{f} \simeq p \circ \hat{g}$.

PROOF: SUPPOSE $\hat{F}: Y \times I \to \hat{X}$ IS THE GIVEN ISOTOPY. THEN $F = p \circ \hat{F}$ IS THE DESIRED ISOTOPY. **DIAGRAM**:

$$\begin{tikzcd} & \hat{X} \arrow[d, "p"] \\ Y \times I \arrow[ur, "\hat{F}"] \arrow[r, "F"'] & X \end{tikzcd}$$

$\square$

THAT IS: "HOMOTOPIES DESCEND".

LEMMA Ⓑ: SUPPOSE $\hat{\alpha}, \hat{\beta}: I \to \tilde{X}$ ARE PATHS WITH $\hat{\alpha}(1) = \hat{\beta}(0)$. THEN $p \circ (\hat{\alpha} * \hat{\beta}) = (p \circ \hat{\alpha}) * (p \circ \hat{\beta})$

PROOF:

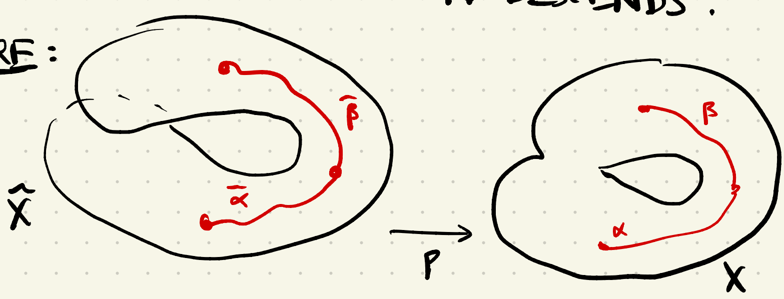
$$(p \circ (\hat{\alpha} * \hat{\beta}))(t) = \begin{cases} (p \circ \hat{\alpha})(2t) & \text{if } t \leq 1/2 \\ (p \circ \hat{\beta})(2t-1) & \text{if } t \geq 1/2 \end{cases}$$

$$= ((p \circ \hat{\alpha}) * (p \circ \hat{\beta}))(t)$$

$\square$

THAT IS "CONCATENATION DESCENDS".

PICTURE:



② FUNDAMENTAL GROUP of S^1.

THEOREM 1.7 $\pi_1(S^1, 1) \cong \mathbb{Z}$

WE FIRST NEED A CANDIDATE HOMOMORPHISM.

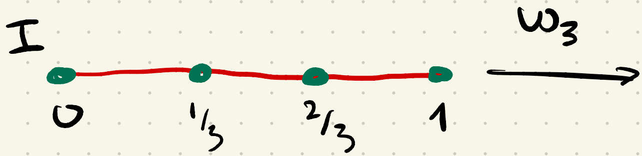
DEFINE FOR $k \in \mathbb{Z}$

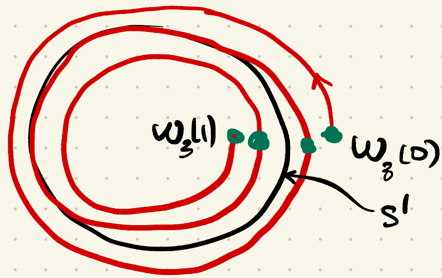
$$\begin{aligned} \omega_k : I &\longrightarrow S^1 \\ t &\longmapsto \exp(2\pi i k t) \end{aligned}$$

WINDS k TIMES ABOUT S^1

NOTE $\omega_0 = e$ IS CONSTANT.

EXAMPLE





DEFINE $\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, 1)$

$$k \longmapsto [\omega_k]$$

PROPOSITION: Φ IS A HOMOMORPHISM.

PROOF: RECALL

$$\begin{aligned} p : \mathbb{R} &\longrightarrow S^1 \\ t &\longmapsto \exp(2\pi i t) \end{aligned}$$

DEFINE:

$$\begin{aligned} \tilde{\omega}_k : I &\longrightarrow \mathbb{R} \\ t &\longmapsto k \cdot t \end{aligned}$$

LOOKS LIKE "CONTINUATION of ω_1"

LEMMA ©: $p \circ \tilde{\omega}_k = \omega_k$

PROOF: $(p \circ \tilde{\omega}_k)(t) = p(kt) = \exp(2\pi i k t) = \omega_k(t)$

□

THAT IS: $\tilde{\omega}_k$ IS A LIFT OF ω_k. [BUT NOT ONLY ONE!

DEF: $\tau_\ell : \mathbb{R} \longrightarrow \mathbb{R}$

$$t \longmapsto t+\ell$$

LEMMA (D): τ_ℓ IS A DECK TRANSFORM, FOR ALL $\ell \in \mathbb{Z}$.

PROOF: NOTE $p \circ \tau_\ell(t) = p(t+\ell)$

$$= \exp(2\pi i t + 2\pi i \ell)$$

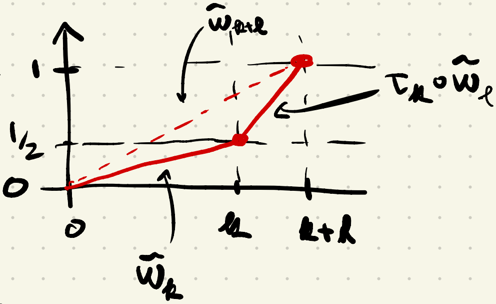
$$= \exp(2\pi i t)$$

$$= p(t)$$

SO $\tau_\ell \in \mathrm{DECK}(p)$. $\square$

LEMMA (E): $\tilde{\omega}_{k+\ell} \underset{\partial}{\simeq} \tilde{\omega}_k * (\tau_k \circ \tilde{\omega}_\ell)$

PICTURE



PROOF: WE USE STRAIGHT-LINE HOMOTOPY IN $\mathbb{R}$.

$$F(s,t) = (1-t)\cdot \tilde{\omega}_{k+\ell}(s) + t\left(\left(\tilde{\omega}_k * (\tau_k \circ \tilde{\omega}_\ell)\right)(s)\right)$$

[CHECK THIS WORKS!] $\square$

WE NOW PROVE THE PROPOSITION VIA COMPUTATION:

$$\Phi(k+l) = [\omega_{k+l}] \qquad \text{DEF } \Phi$$

$$= [p \circ \hat{\omega}_{k+l}] \qquad \text{LEMMA (C)}$$

$$= [p \circ (\hat{\omega}_k * (\tau_k \circ \hat{\omega}_l))] \qquad \text{LEMMAS (E)} + \text{(A)}$$

$$= [(p \circ \hat{\omega}_k) * (p \circ \tau_k \circ \hat{\omega}_l)] \qquad \text{LEMMA (B)}$$

$$= [p \circ \hat{\omega}_k] \cdot [p \circ \hat{\omega}_l] \qquad \text{LEMMA (D)}$$

$$= [\omega_k][\omega_l] \qquad \text{LEMMA (C)}$$

$$= \Phi(k)\,\Phi(l) \qquad \text{DEF } \Phi. \qquad \square.$$

PROP.

NEXT TWO LECTURES. Φ IS AN ISOM.