2024-10-21 LECTURE 10 MA3F1 SAUL SCHLEIMER

PROP 1.30: COVERING SPACES HAVE THE HOMOTOPY LIFTING PROPERTY.

SUPPOSE: $p: \tilde{X} \to X$, $F: Y \times I \to X$, $g: Y \times \{0\} \to \tilde{X}$ ALL GIVEN. [WITH $F_0 = p \circ G$]. WE MUST BUILD $\tilde{F}: Y \times I \to \tilde{X}$ WITH $p \circ \tilde{F} = F$, $\tilde{F}_0 = g$

DIAGRAM:

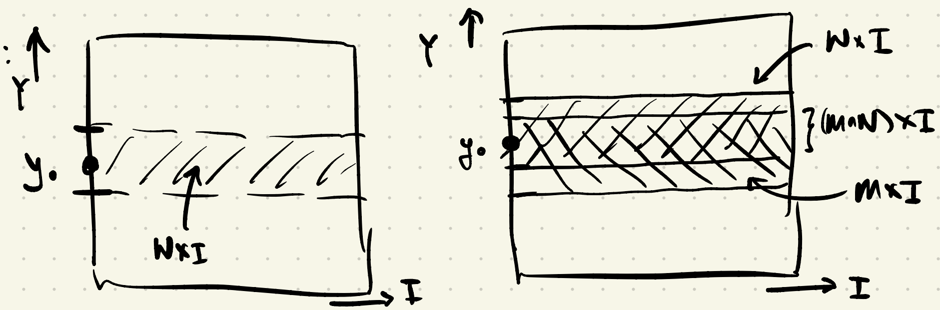
$$\begin{tikzcd} Y \times \{0\} \arrow[r, "g"] \arrow[d] & \tilde{X} \arrow[d, "p"] \\ Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"] & X \end{tikzcd}$$

WE WILL BUILD $\tilde{F}$ OUT OF PIECES OF F.

CLAIM: FOR ANY $y_0 \in Y$ THERE EXISTS AN OPEN SET $N \subset Y$ WITH $y_0 \in N$ AND THERE EXISTS (A) A UNIQUE (B) $\hat{F}_N : N \times I \to \tilde{X}$ SO THAT (i) $(\hat{F}_N)_0 = g|N$ AND (ii) $p \circ \hat{F}_N = F|N \times I$.

FURTHERMORE IF $y \in M$ (OPEN) AND $\hat{F}_M$ IS ANOTHER SUCH THEN $\hat{F}_M|(M \cap N) \times I = \hat{F}_N|(M \cap N) \times I$ (C)

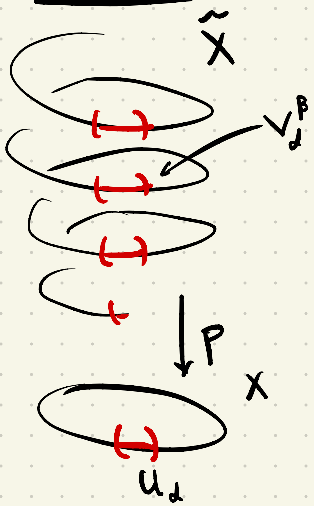
PICTURE:



THE CLAIM ALLOWS US TO COVER Y BY SUCH OPEN SETS $\{N\}$. WE FORM $\tilde{F} = \bigcup_N \tilde{F}_N$ AND APPLY THE GLUING LEMMA. SO CLAIM $\Rightarrow$ (1.30).

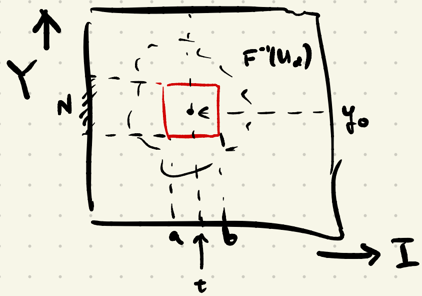
RECALL $p : \tilde{X} \to X$ IS A COVERING MAP.
LET $\{U_\alpha\}$ BE THE GIVEN OPEN COVER OF X.
LET $\{V_\alpha^\beta\}$ BE THE GIVEN OPEN PARTITION of $p^{-1}(U_\alpha)$.
LET $g_\alpha^\beta : U_\alpha \to V_\alpha^\beta$ BE THE HOMEOMORPHISM INVERSE TO $p|V_\alpha^\beta$.

PICTURE



FOR ANY $t \in I$ THERE IS SOME U_α, SO THAT $(y_0, t) \in U_\alpha$. SINCE F IS CONTINUOUS THE SET $F^{-1}(U_\alpha)$ IS OPEN. SO THERE IS SOME $N_t \subset Y$, $I_t \subset I$ OPEN WITH $N_t \times I_t \subset F^{-1}(U_\alpha)$

PICTURE

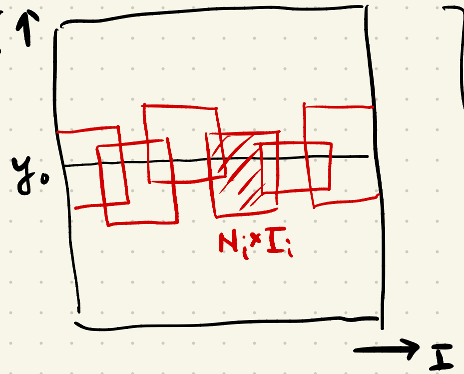


THUS: $F(N \times (a,b)) \subset U_\alpha$, AND

$$(g_\alpha^\beta \circ F)(N \times (a,b)) \subset V_\alpha^\beta.$$

NOTE THE SETS $\{N_t \times I_t\}_{t \in I}$ IS AN OPEN COVER of $\{y_0\} \times I$. SINCE $\{y_0\} \times I$ IS COMPACT THERE IS A FINITE SUBCOVER $\{N_i \times I_i\}_{i=0}^{n}$

PICTURE



SET $N = \cap N_i$;
SO N OPEN, $y_0 \in N$.
ALSO (BY THE LEBESGUE COVERING LEMMA) THERE IS A PARTITION

$0 = t_0 < t_1 < \cdots < t_i < \cdots < t_n = 1$ of I SO THAT,
FOR ALL i THERE IS SOME U_α WITH $F(N \times [t_i, t_{i+1}]) \subset U_\alpha$.
WE NOW DEFINE $\tilde{F}_N^k : N \times [0, t_k] \longrightarrow \tilde{X}$ BY RECURSION.
TO HAVE THE PROPERTY $p \circ \tilde{F}_N^k = F|N \times [0, t_k]$.

BASE CASE $k = 0$

DEFINE $\tilde{F}_N^0 : N \times [0] \longrightarrow \tilde{X}$ BY $\tilde{F}_N^0(y, 0) = g(y, 0)$ ✓

HERE-ISH

INDUCTION STEP:

WE SUPPOSE $\tilde{F}_N^k : N \times [0, t_k] \longrightarrow \tilde{X}$ IS DEFINED.

NOTE $F(N \times [t_k, t_{k+1}]) \subset U_\alpha$ FOR SOME α.

AFTER SHRINKING N (TO GET SMALLER NEIGH. of y_0) THERE

IS SOME β SO THAT $F(N \times \{t_k\}) \subset V_\alpha^\beta$.

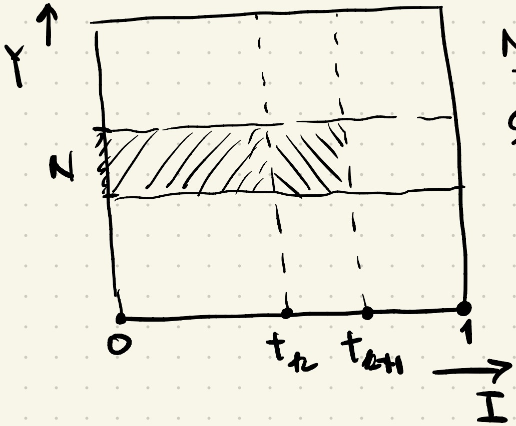
REPLACE $N \times \{t_k\}$ BY INTERSECTION WITH $(\tilde{F}_N^k)^{-1}(V_\alpha^\beta)$.

SO DEFINE

$$\tilde{F}_N^{k+1}(z, t) = \begin{cases} \tilde{F}_N^k(y, t), & \text{if } t \le t_k \\ (g_\alpha^\beta \circ F)(y, t), & \text{if } t \geq t_k \end{cases}$$

NOTE $\tilde{F}_N^{k+1}$ IS CONTINUOUS BY THE GLUING LEMMA

PICTURE:



NOTE: $p \circ g^{\beta}_{\alpha} = Id_{U_\alpha}$

SO: $p \circ \hat{F}_N^{k+1} = F|N \times [0, t_{k+1}]$

SUPPOSE $\hat{F}_N$ AND $\hat{F}_N'$ ARE DEFINED THIS WAY

WE MUST SHOW $\hat{F}_N = \hat{F}_N'$. SO FIX $z \in N$.
CHOOSE PARTITION $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m = 1$
SO FOR ALL j THERE IS α WITH

$$F(\{z\} \times [t_j, t_{j+1}]) \subset U_\alpha.$$

NOW: $\hat{F}_N(z,0) = \hat{F}_N'(z,0)$ BECAUSE BOTH AGREE WITH $g(z,0)$.

INDUCTION: SUPPOSE $\hat{F}_N(z,t) = \hat{F}_N'(z,t)$ FOR $z \in [0, t_k]$.

SINCE $[t_k, t_{k+1}]$ IS CONNECTED SO IS $\hat{F}_N(\{z\} \times [t_k, t_{k+1}])$ AND $\hat{F}_N'(\{z\} \times [t_k, t_{k+1}])$. SO THESE ARE CONTAINED IN V_α^β AND $V_\alpha^{\beta'}$ FOR UNIQUE CHOICES OF β, β'. BUT $\hat{F}_N(z, t_k) = \hat{F}_N'(z, t_k)$ SO $\beta = \beta'$.

FINALLY: THE SAME PROOF SHOWS THAT (1.30)

$$\hat{F}_M | (M \cap N) \times I = \hat{F}_N | (M \cap N) \times I. \text{ SO } \hat{F} = \bigcup_N \hat{F}_N \qquad \square$$