2024-10-31 MA3F1 SAUL SCHLEIMER LECTURE 15

① COROLLARIES of NO RETRACT THM.

TABLECLOTH THEOREM: SUPPOSE $f : \mathbb{D}^2 \to \mathbb{D}^2$ HAS $f|S^1 = \mathrm{Id}_{S^1}$. THEN $f$ IS SURJECTIVE.

PERRON-FROBENIUS $(n=3)$: SUPPOSE $A \in \mathrm{MAT}_{3,3}(\mathbb{R})$ HAS $a_{ij} > 0$ FOR ALL $i,j$. THEN $A$ HAS AN EIGENVALUE $\lambda > 0$ AND A $\lambda$-EIGENVECTOR $v$ WITH $v_i > 0$ FOR ALL $i$.

THE NEXT RESULT REQUIRES MORE WORK TO PROVE

FUNDAMENTAL THM of ALGEBRA SUPPOSE $Q \in \mathbb{C}[z]$ IS A NON-CONSTANT POLYNOMIAL. THEN THERE IS SOME $z_0 \in \mathbb{C}$ SO THAT $Q(z_0) = 0$. [THAT IS, Q HAS A ROOT.]

② NULL HOMOTOPIES:

SUPPOSE $f : X \to Y$ IS GIVEN. WE SAY $f$ IS NULL-HOMOTOPIC IF $f \simeq e$ [A CONSTANT MAP]

DEF: A POINTED MAP $f : (X, x_0) \to (Y, y_0)$ IS NULL HOMOTOPIC REL BASEPOINT IF THERE IS A HOMOTOPY $F : X \times I \to Y$ WITH $f_0 = f$, $f_1 = e$ $\left[\begin{array}{l} e(x) = y_0 \text{ FOR} \\ \text{ALL } x \in X \end{array}\right]$, AND $F(x_0, t) = y_0$.

THAT IS $f \underset{x_0}{\simeq} e$.

PROPOSITION: FOR $X = S^1$ THE FIRST IMPLIES THE SECOND.

[THIS IS NOT THE CASE FOR GENERAL X  ]

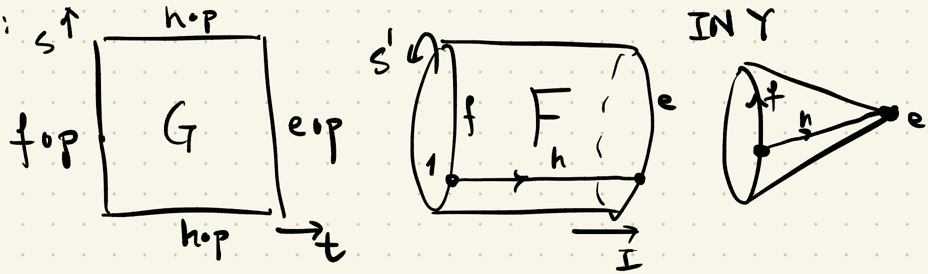
PROOF: SUPPOSE $F : S^1 \times I \to Y$ IS THE GIVEN HOMOTOPY. DEFINE $G : I \times I \to Y$ BY

$$G(s,t) = F(p(s), t)$$

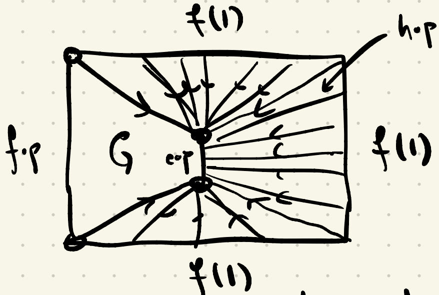
FOR $p(s) = \exp(2\pi i s)$

DEF $h : I \to Y$ BY $h(t) = F(1, t)$.

PICTURE:




SO BUILD $H: I\times I \longrightarrow Y$ BY




FINALLY DEFINE

$K: S^1\times I \longrightarrow Y$

$(z,t) \longmapsto G\left(\frac{\log(z)}{2\pi i}, t\right)$

THIS IS THE DESIRED POINTED HOMOTOPY.

DEFINE $p_k: S^1 \longrightarrow S^1$ BY $p_k(z) = z^k$.

<u>COROLLARY</u>: $p_k \simeq p_\ell$ IFF $k=\ell$. $\square$

<u>SO</u>: $p_k$ IS NULL HOMOTOPIC IFF $k=0$.

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## ③ <u>PROOF SKETCH of FUND. THM of ALGEBRA.</u>

SUPPOSE, FOR A CONTRADICTION, THAT HAS NO ROOT.
SO $Q(\mathbb{C}) \subset \mathbb{C} - \{0\}$. FOR $R \in \mathbb{R}_{>0}$ DEFINE

$$C_R = \{z\in\mathbb{C} \mid |z| = R\}$$

SUPPOSE $Q(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, WITH $a_n \neq 0$. SINCE IT DOES NOT CHANGE THE ROOTS WE MAY DIVIDE AND SO ASSUME $a_n = 1$.

NOW PICK $R >> 0$ SO THAT $R^n >> \sum_{k=0}^{n-1} |a_k| R^k$

WITH THIS CHOICE of $R$

$$Q|C_R : C_R \longrightarrow \mathbb{C}-\{0\}$$

IS HOMOTOPIC TO

$$q: C_R \longrightarrow \mathbb{C}-\{0\}$$
$$z \longmapsto z^n$$

IN FACT BY STRAIGHT-LINE HOMOTOPY.

SO: $q$ IS HOMOTOPIC TO $p_n : S^1 \longrightarrow \mathbb{C}-\{0\}$

$$z \longmapsto z^n$$

SO $Q|C_R$ IS NOT NULL-HOMOTOPIC IN $\mathbb{C}-\{0\}$.

NOW CONSIDER

$$F: C_R \times I \longrightarrow \mathbb{C}-\{0\}$$
$$(z,t) \longmapsto Q(t\cdot z)$$

THIS IS A NULL-HOMOTOPY of $Q|C_R \times \{0\}$.

THIS IS THE DESIRED CONTRADICTION □

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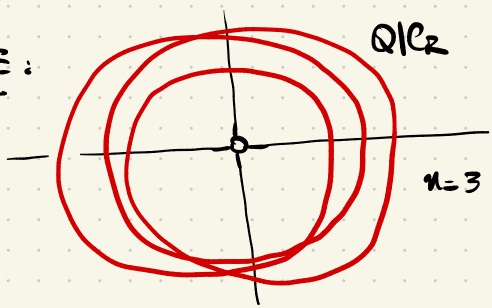
PICTURE:

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④ EVEN/ODD. SUPPOSE $X,Y$ ARE SUBSETS of VECTOR SPACES. SUPPOSE $X,Y$ ARE INVARIANT UNDER NEGATION.

DEF: SAY $f: X \to Y$ IS

EVEN } if { $f(-x) = f(x)$

ODD } if { $f(-x) = -f(x)$

**EXAMPLES**: (1) $\sin : \mathbb{R}^1 \to \mathbb{R}^1$ IS ODD

(2) $\cos : \mathbb{R}^1 \to \mathbb{R}^1$ IS EVEN

(3) $\mathrm{Id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ IS ODD.

(4) $$\left.\begin{aligned} P_k : S^1 &\longrightarrow S^1 \\ z &\longmapsto z^k \end{aligned}\right\} \text{ IS } \begin{Bmatrix} \text{EVEN} \\ \text{ODD} \end{Bmatrix} \text{ IF } k \equiv \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \bmod 2$$

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