2024-11-04 SAUL SCHLEIMER LECTURE 16 MA3F1

(1) EVEN AND ODD

PROP (A) SUPPOSE $f: S^1 \to S^1$ IS ODD. THEN f IS NOT NULL HOMOTOPIC.

PROOF of (A): SUPPOSE $f: S^1 \to S^1$ IS ODD. SUPPOSE $f \simeq e$ VIA $F: S^1 \times I \to S^1$. DEFINE $G: S^1 \times I \to S^1$, $(z,t) \mapsto \frac{F(z,t)}{F(1,t)}$

FOR A CONTRADICTION

NOTE THAT $G(1,t) = \frac{F(1,t)}{F(1,t)} = 1$.

SO G IS A POINTED HOMOTOPY.

NOTE $g = g_0 : S^1 \to S^1$ IS AGAIN ODD. [g IS A ROTATION of f]

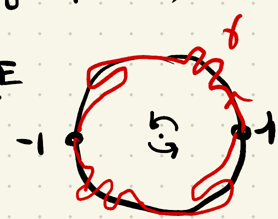
DEFINE $p: \mathbb{R} \to S^1$ BY $p(s) = \exp(2\pi i s)$.

DEFINE $\gamma: I \to S^1$ BY $\gamma = g \circ p$, SO $\gamma(s) = g(\exp(2\pi i s))$.

SINCE g IS ODD WE FIND:

$\gamma(t + 1/2) = -\gamma(t)$ FOR $t \in [0, 1/2]$.

PICTURE



SINCE $\gamma(0) = \gamma(1) = 1$ DEDUCE $\gamma(1/2) = -1$.

NOW LIFT γ to $\hat{\gamma}: I \to \mathbb{R}$ WITH $\hat{\gamma}(0) = 0$.

SO $\hat{\gamma}(1/2) = n + 1/2$ FOR SOME $n \in \mathbb{Z}$ [B/C $\gamma(1/2) = -1$].

DEFINE $\tau_{n+1}: \mathbb{R} \to \mathbb{R}$ BY $t \mapsto t + n + 1/2$.

DEFINE $\delta : I \to S^1$ BY $\delta(t) = \gamma(t/2)$
$\varepsilon : I \to S^1$ BY $\varepsilon(t) = \gamma(t/2 + 1/2)$ } SO $\gamma = \delta * \varepsilon$ AND $\varepsilon(t) = -\delta(t)$

DEFINE $\tilde{\delta} : I \to S^1$ AND $\hat{\varepsilon} : I \to S^1$ TO BE THE LIFTS of δ AND ε WITH $\tilde{\delta}(0) = 0$ AND $\tilde{\varepsilon}(0) = n + 1/2$.

BY UNIQUENESS of LIFTING $\tilde{\gamma} = \tilde{\delta} * \tilde{\varepsilon}$.

CLAIM: $\hat{\varepsilon} = \tau_{n+1/2} \circ \tilde{\delta}$.

PROOF: WE USE UNIQUENESS.

FIRST, $\hat{\varepsilon}(0) = n + 1/2$ AND $(\tau_{n+1/2} \circ \tilde{\delta})(0) = \tau_{n+1/2}(\tilde{\delta}(0)) = \tau_{n+1/2}(0) = n + 1/2$.

NEXT:
$$\begin{aligned}
(p \circ \tau_{n+1/2} \circ \hat{\delta})(t) &= p(\tau_{n+1/2}(\tilde{\delta}(t))) \\
&= p(n + 1/2 + \hat{\delta}(t)) \\
&= \exp(2\pi i \cdot (n + 1/2 + \tilde{\delta}(t))) \\
&= \exp(2\pi i n + \pi i + 2\pi i \hat{\delta}(t)) \\
&= -1 \cdot \exp(2\pi i \tilde{\delta}(t)) \\
&= -\delta(t) \\
&= \varepsilon(t)
\end{aligned}$$

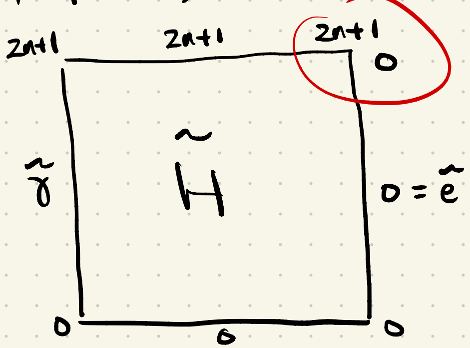
So $\tau_{n+1/2} \circ \hat{\delta}$ IS A LIFT of ε, THUS EQUALS $\hat{\varepsilon}$. □

THUS $\hat{\gamma}(1) = \hat{\varepsilon}(1) = n + 1/2 + \tilde{\delta}(1) = 2n+1$, WHICH IS ODD.

DEFINE $H : I \times I \to S^1$ BY $H(s,t) = G(p(s), t)$.

SO $h_0 = \gamma$, $h_1 = e$, AND H IS A HOMOTOPY REL ENDPTS.

NOW LIFT:



} SO $2n+1 = 0$ AND 0 IS ODD. $\Rightarrow\!\Leftarrow$

□

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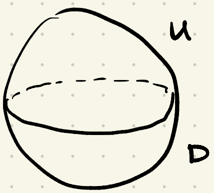
PROP B: SUPPOSE $f: S^2 \to \mathbb{R}^2$ IS ODD. THEN $f(x)=0$ FOR SOME $x \in S^2$.

PROOF of B SUPPOSE $f(x)$ NEVER VANISHES.

DEFINE $g: S^2 \longrightarrow S^1$, $x \longmapsto f(x)/|f(x)|$ } SO g IS ALSO ODD

DEFINE $U \subset S^2$ BY $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1,\ z \geq 0\}$



DEFINE $\partial U = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=1,\ z=0\}$

SO $g|\partial U$ IS ODD AND NOT NULL HOMOTOPIC BY A.

BUT $g|U$ IS A NULL HOMOTOPY of $g|\partial U$;

GIVING THE DESIRED CONTRADICTION □

3 **BORSUK-ULAM THEOREM.**

THEOREM 1.10: SUPPOSE $f: S^2 \longrightarrow \mathbb{R}^2$ IS ANY MAP. THEN THERE IS SOME $x \in S^2$ WITH $f(-x) = f(x)$.

PROOF GIVEN $f: S^2 \longrightarrow \mathbb{R}^2$ ANY MAP. DEFINE $g: S^2 \to \mathbb{R}^2$ BY $g(x) = f(x) - f(-x)$ THIS IS ODD, SO HAS A ROOT, BY B □

"AT ANY MOMENT THERE ARE ANTIPODES WITH THE SAME TEMPERATURE AND HUMIDITY."

THE "HAM SANDWICH THEOREM". GIVEN $U, V \subset \mathbb{R}^3$ MEASURABLE SETS, THERE IS A PLANE CUTTING BOTH IN HALF

PICTURE