2024-11-05 LECTURE 17 MA3F1 SAUL SCHLEIMER

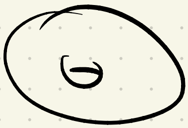
1 TORI AND SPHERES.

THEOREM 1.12: SUPPOSE (X, x_0), (Y, y_0) ARE POINTED SPACES. THEN
$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

PROOF: DEFINE $p_X : X \times Y \longrightarrow X$, $(x, y) \longmapsto x$ AND $p_Y : X \times Y \longrightarrow Y$, $(x, y) \longmapsto y$

AND CONSIDER $(p_X)_* \times (p_Y)_*$ $\square$

EXAMPLES: $\mathbb{T}^2 = S^1 \times S^1$

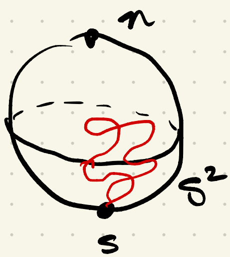


HAS $\pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2$

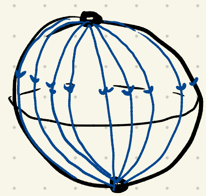
AND $\pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$.

THEOREM 1.14: SUPPOSE $n \geq 2$. THEN $\pi_1(S^n) \cong \mathbb{1}$.

PROOF: SET $s = (0, 0, \ldots, -1)$, $n = (0, 0, \ldots, 1)$.



SUPPOSE $\gamma \in \text{LOOPS}(S^n, s)$. HOMOTOPE γ OFF of n. THEN HOMOTOPE ALONG LINES of LONGITUDE



$\square$

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2 THE GALOIS CORRESPONDENCE

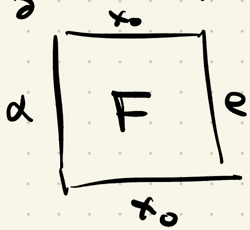
THEOREM: SUPPOSE (X, x_0) IS A "NICE" POINTED SPACE. THEN THERE IS A "NATURAL" BIJECTION
$$\left\{ \begin{array}{c} \text{SUBGROUPS of} \\ \pi_1(X, x_0) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{POINTED} \\ \text{CONNECTED} \\ \text{COVERS of } X \end{array} \right\} \Big/ \text{ISOMORPHISM}$$

HERE IS THE BACKWARDS MAP:

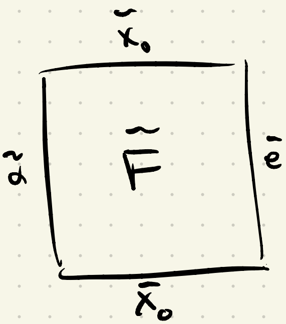
PROPOSITION (1.31) SUPPOSE $p:(\tilde{X},\tilde{x}_0) \longrightarrow (X,x_0)$ IS A PATH-CONNECTED POINTED COVER. THEN p_* IS INJECTIVE. FURTHERMORE: SUPPOSE $\alpha \in \text{LOOPS}(X,x_0)$ AND $\hat{\alpha}: I \longrightarrow \tilde{X}$ IS THE LIFT of α WITH $\hat{\alpha}(0)=\tilde{x}_0$. THEN $\hat{\alpha}$ IS A LOOP IF AND ONLY IF

$$[\alpha] \in p_*\left(\pi_1(\tilde{X},\tilde{x}_0)\right) = \text{IMAGE}(p_*).$$

PROOF: FIX $[\tilde{\alpha}] \in \pi_1(\tilde{X},\tilde{x}_0)$ WITH $p_*([\alpha]) = [e]$. DEFINE $\alpha = p \circ \tilde{\alpha}$. SO $\alpha \underset{\partial}{\simeq} e$. SUPPOSE $F: I \times I \longrightarrow X$ IS THE GIVEN HOMOTOPY.



WE LIFT TO OBTAIN $\hat{F}: I \times I \longrightarrow \tilde{X}$



BECAUSE CONSTANT PATHS LIFT TO CONSTANT PATHS.

THUS $\tilde{\alpha} \underset{\partial}{\simeq} \hat{e}$ SO $[\tilde{\alpha}] = [\hat{e}]$ AND p_* IS INJECTIVE.

EXERCISE: PROVE THE "FURTHERMORE" ∎

(3) INDEX vs DEGREE

RECALL: SUPPOSE G IS A GROUP. SUPPOSE $H < G$ THEN $[G:H]$, THE INDEX of H IN G, IS THE CARDINALITY of

$$H\backslash G = \{H \cdot g \mid g \in G\} = \text{SET of RIGHT COSETS}.$$

DEF: SUPPOSE $p:(\tilde{X},\tilde{x}_0) \longrightarrow (X,x_0)$ IS A PATH-CONNECTED

COVERING SPACE. THE $\deg(p)$ (THE DEGREE of p) IS $\mathrm{CARD}(p^{-1}(x_0))$.

EXERCISE: $\mathrm{CARD}(p^{-1}(x)) = \deg(p)$ FOR ALL $x \in X$.

PROP (1.32) SUPPOSE $p:(\tilde X,\tilde x_0)\longrightarrow(X,x_0)$ IS A PATH-CONN COVER. THEN

$$\deg(p) = [\pi_1(X,x_0) : p_*(\pi_1(\tilde X,\tilde x_0))]$$

AXIOM of CHOICE IF $\deg(p)$ IS INF.

PROOF: SET $p^{-1}(x_0)=\{\tilde x_i\}_i$. PICK PATHS

$\tilde\beta_i : I\longrightarrow\tilde X$ WITH $\tilde\beta_i(0)=\tilde x_0$ AND $\tilde\beta_i(1)=\tilde x_i$

SO $\beta_i = p\circ\tilde\beta_i \in \mathrm{LOOPS}(X,x_0)$.

SET $G=\pi_1(X,x_0)$ AND $H=p_*(\pi_1(\tilde X,\tilde x_0))$.

BY (1.31) $[\beta_i]\in H$ IFF $i=0$.

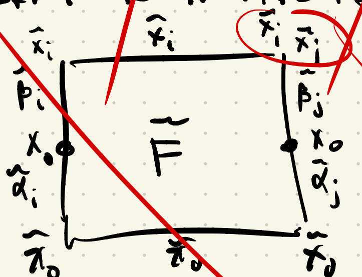
CLAIM ①: $H[\beta_i]=H[\beta_j]$ IFF $i=j$.

CLAIM ②: FOR ALL $g\in G$ THERE IS i SO THAT $g\in H[\beta_i]$

PROOF ①: SUPPOSE $H[\beta_i]=H[\beta_j]$ SO THERE IS SOME $\tilde\alpha_i,\tilde\alpha_j\in\mathrm{LOOPS}(\tilde X,\tilde x_0)$ WITH

$$(p\circ\tilde\alpha_i)*\beta_i \underset{\partial}{\simeq} (p\circ\tilde\alpha_j)*\beta_j$$

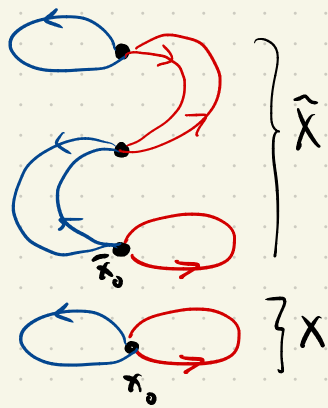
SO LIFT TO $\tilde X$ AND FIND



THUS $i=j$ AS DESIRED ①

~~PROOF of ②: FIX ANY $[\alpha] \in G$. LIFT TO $\tilde{\alpha}: I \to \hat{X}$~~
~~WITH $\tilde{\alpha}(0) = \tilde{x}_0$. SUPPOSE $\tilde{\alpha}(1) = \tilde{x}_i$. SO~~
~~$\tilde{\alpha} * \overline{\tilde{\beta}_i} \in$ LOOPS$(\hat{X}, \tilde{x}_0)$. THUS $[p \circ (\tilde{\alpha} * \overline{\tilde{\beta}_i})] \in H$~~
~~THAT IS $[\alpha]\cdot[\beta_i]^{-1} \in H$ SO $[\alpha] \in H\cdot[\beta_i]$~~ ②
~~THUS $\tilde{x}_i \longrightarrow H\cdot[\beta_i]$ IS A BIJECTION~~
~~FROM $p^{-1}(x_0)$ TO $H\backslash G$~~ □

EXAMPLE:
WITH DEG=3.



TO UNDERSTAND THIS WE MUST UNDERSTAND $\pi_1(X, x_0)$ AND, MORE GENERALLY, THE FUNDAMENTAL GROUPS of GRAPHS.