2024-11-11 SAUL SCHLEIMER MA3F1 LECTURE 19

① FREE PRODUCTS LET $\{G_\alpha\}_\alpha$ BE A COLLECTION of GROUPS.
LET W BE THE SET of WORDS OVER $\{G_\alpha\}$.

LEMMA: EVERY $w \in W$ HAS A UNIQUE REDUCTION.

PROOF: INDUCT ON LENGTH. [EXERCISE] $\square$

LET $R = \{w \in W \mid w \text{ IS REDUCED}\}$.

DEFINE $r: W \longrightarrow R$ WITH $r(w)$ THE REDUCTION of w.

RECALL FOR $w = [g_1, g_2, \cdots, g_n]$

WE DEFINE $\bar{w} = [g_n^{-1}, g_{n-1}^{-1}, \cdots, g_1^{-1}]$.

WE DEFINE, FOR $u, v \in R$, $\boxed{u \cdot v = r(u * v)}$.

NOW WE DEFINE $\underset{\alpha}{*} G_\alpha = (R, \cdot)$

THEOREM $\underset{\alpha}{*} G_\alpha$ IS A GROUP.

PROOF: (i) $\varepsilon = [\,]$ IS THE IDENTITY

BECAUSE $u \cdot \varepsilon = r(u * \varepsilon) = r(u) = u$

AND SIMILARLY $\varepsilon \cdot u = u$.

(ii) $u^{-1} = \bar{u}$ BECAUSE $u \cdot u^{-1} = r(u * \bar{u}) = r(\varepsilon) = \varepsilon$

AND SIMILARLY $u^{-1} \cdot u = \varepsilon$.

(iii) ASSOCIATIVITY IS MORE DIFFICULT!

WE USE A TECHNIQUE DUE TO VAN DER WAERDEN

RECALL $\mathrm{SYM}(R)$ IS THE GROUP of BIJECTIONS of R.

FOR ALL α, FOR ALL $g \in G_\alpha - \{\varepsilon_\alpha\}$, $L_g : R \longrightarrow R$

$u \longmapsto [g] \cdot u$

ALSO DEFINE $L_\varepsilon = \mathrm{Id}_R$ FOR $\varepsilon \in G_\alpha$

CLAIM: FOR $g, h \in G_\alpha$, $L_g \circ L_h = L_{g \cdot h}$ $[g \cdot h \in G_\alpha]$

PROOF: LET $u = [g_1, \cdots, g_n]$.

CASE (1) $g_1 \notin G_\alpha$.

CASE (1i) $g \neq h^{-1}$.

THEN $L_g \circ L_h(u) = [g \cdot h, g_1, g_2, \cdots g_n] = L_{gh}(u)$. ✓

SKIP

CASE (1ii) $g = h^{-1}$

THEN $L_g \circ L_h(u) = [g_1, g_2, \cdots g_n] = u = L_\varepsilon(u)$ ✓

CASE (2) $g_1 \in G_\alpha$.

CASE (2i): $|u| = 1$. **CASE (2ia)** $g \cdot h \cdot g_1 \neq \varepsilon$.

THEN $L_g \circ L_h(u) = [g \cdot h \cdot g_1] = L_{gh}(u)$

CASE (2ib) $g \cdot h \cdot g_1 = \varepsilon$. THEN $L_g \circ L_h(u) = [\,] = L_{gh}(u)$

CASE (2ii) $|u| > 1$. SO $g_2 \notin G_\alpha$.

CASE (2iia) $g h \cdot g_1 \neq \varepsilon$ SO $L_g \circ L_h(u) = [ghg_1, g_2, \cdots g_n]$

$= L_{gh}(u)$.

CASE (2iib) $g \cdot h \cdot g_1 = \varepsilon$ SO $L_g \circ L_h(u) = [g_2, g_3, \cdots, g_n]$

$= L_{gh}(u)$.

CLAIM: $L_{g^{-1}} = (L_g)^{-1}$

PROOF: $L_g \circ L_{g^{-1}} = L_\varepsilon = Id_R$ $\square$

THUS L_g IS BIJECTIVE.

DEFINE: $L : R \longrightarrow SYM(R)$

$[g_1, g_2, \cdots g_n] \longmapsto L_{g_1} \circ L_{g_2} \circ L_{g_3} \circ \cdots \circ L_{g_n}$

NOTATION: $L(u) = L_u$

CLAIM: $L : R \longrightarrow \mathrm{Sym}(R)$ IS INJECTIVE.

PROOF: $L_u(\varepsilon) = u$.

SO $L_v = L_u$ IF AND ONLY IF $u = v$ $\square$

CLAIM: FOR $u, v \in R$, $L_u \circ L_v = L_{u \cdot v}$

PROOF: DEFINE $w \in R$ MAXIMAL SO THAT

$$\begin{cases} u = u' * w \\ v = \bar{w} * v' \end{cases}$$

SUBCLAIM: $u \cdot v = u' \cdot v'$ [BUT NOT NEC. EQUAL TO $u' * v'$]

PROOF: INDUCT ON LENGTH of w. $\square$

NOW $\left.\begin{matrix} L_u = L_{u'} \circ L_w \\ L_v = L_{\bar{w}} \circ L_{v'} \end{matrix}\right\}$ SO $L_u \circ L_v = (L_{u'} \circ L_w) \circ (L_{\bar{w}} \circ L_{v'})$

$$= L_{u'} (L_w L_{\bar{w}}) \circ L_{v'}$$

$$= L_{u'} \circ Id_R \circ L_{v'}$$

$$= L_{u'} \circ L_{v'}$$

TWO CASES AS $u' \cdot v' = u' * v'$ OR NOT.

$$= L_{u' \cdot v'}$$

$$= L_{u \cdot v} \quad \square$$

SKIP

CLAIM: SUPPOSE $u, v, w \in R$. THEN $(u \cdot v) \cdot w = u \cdot (v \cdot w)$

PROOF: $L_{(u \cdot v) \cdot w} = L_{u \cdot v} \circ L_w$

$$= (L_u \circ L_v) \circ L_w$$

$$= L_u \circ (L_v \circ L_w)$$

$$= L_u \circ L_{v \cdot w} = L_{u \cdot (v \cdot w)}.$$

BUT $L : R \longrightarrow \mathrm{SYM}(R)$ IS INJECTIVE. $\square$.

THIS COMPLETES THE PROOF OF ASSOCIATIVITY. $\square$.

<u>SUMMARY</u>: WE "IMPORT" ASSOC. FROM ONE GROUP $(\mathrm{SYM}(R))$ TO ANOTHER $(\ast_\alpha G_\alpha)$!

SKIP.

THERE IS A MORE TOPOLOGICAL PROOF, USING TREES [GRAPHS WITHOUT CYCLES] AND THE FOLLOWING

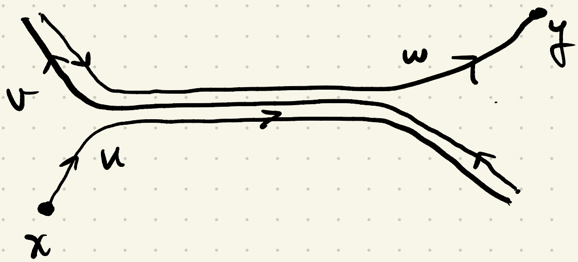
<u>LEMMA</u>: ANY TWO POINTS IN A TREE ARE CONNECTED BY A UNIQUE EMBEDDED PATH.

SUPPOSE u, v, w ARE REDUCED WORDS.

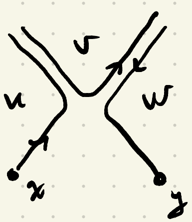
(1) BUILD A GRAPH $G(u,v,w)$ BY GLUING INTERVALS I_u, I_v, I_w ALONG CANCELLING SUBINTERVALS. MARK FIRST POINT x of I_u, LAST PT y of I_w.

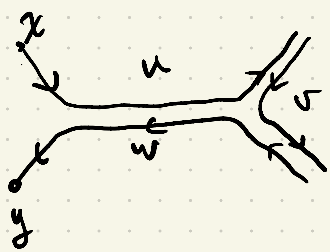
(2) PROVE $G(u,v,w)$ IS A TREE.

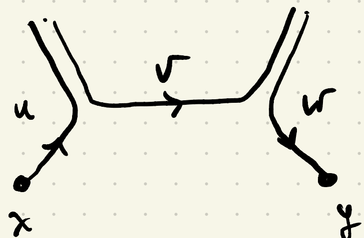
(3) READ OFF WORD BETWEEN x, y IN $G(u,v,w)$ TO GET $u \cdot v \cdot w$.



AND MANY MORE CASES: [NUMBER of LEAVES, WHICH ARCS CROSS BRIDGE [IF EXISTS] - - -]







ETC.