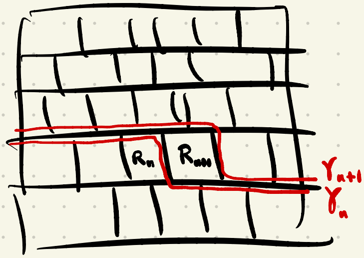
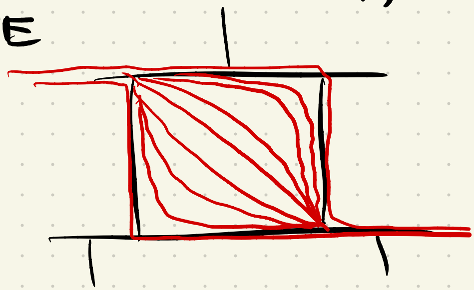
2024-11-18 LECTURE 22 MA3F1 SAUL SCHLEIMER

① FINISH 1.20: WE HAVE TILED $I \times I$ WITH RECTANGLES $\{R_i\}$ SO THAT FOR ALL i THERE IS $\alpha(i)$ WITH $G(R_i) \subset A_{\alpha(i)}$ PICTURE



WE APPLY MOVES (REDUCTIONS / EXPANSIONS / EXCHANGES) UNTIL $G(\gamma_n - \gamma_{n+1})$ LIES IN $A_{\alpha(n+1)}$.

NOW HOMOTOPE



THIS DOES NOT CHANGE THE HOMOTOPY CLASSES of THE FACTORS BUT DOES CHANGE THE CHOSEN REPRESENATIVES. FINALLY

$$\left\{ \begin{matrix} [g_0] \\ [g_1] \end{matrix} \right\} \text{ CONNECT TO } \left\{ \begin{matrix} [G \mid \gamma_0] \\ [G \mid \gamma_N] \end{matrix} \right\}$$

VIA SEQUENCES of MOVES.

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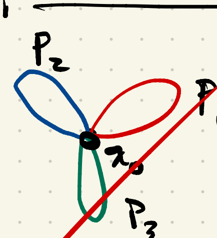
② WHERE EXAMPLES COME FROM:

WE DEFINE $R_n = \bigvee_n S^1$

$\underline{R_3}$:

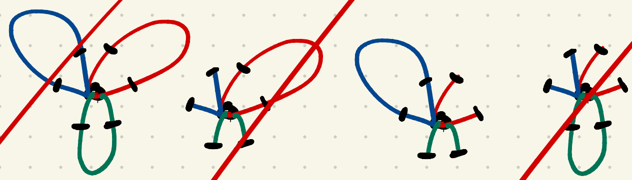


LET $U \subset R_n$ BE A SMALL NEIGH. of $x_0 \in R_n$. NOTE $U \underset{h.e.}{\simeq} \{pt\}$ SO $\pi_1(U, x_0) \cong 1$

[EXERCISE] LET P_i BE THE i^{th} PETAL of R_n.

SO $P_i \cong S^1$. ALSO $A_i = P_i \cup U \underset{h.e.}{\simeq} S^1$.

<u>PICTURE:</u>



SO $\pi_1(A_i) \cong \mathbb{Z}$ AND $\pi_1(A_i \cap A_j) \cong 1$.

SO $\pi_1(R_n) \cong \ast_n \mathbb{Z}$ <u>DEF</u>: THIS IS THE <u>FREE GROUP</u> of RANK n.

<u>EXERCISE</u>: $\mathbb{C} - \{n \text{ points}\} \underset{h.e.}{\simeq} R_n$ SO $\pi_1(\mathbb{C} - \{n \text{ pts}\}) \cong F_n$.

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(3) <u>CW COMPLEXES</u> (C = CLOSURE FINITE, W = WEAK TOPOLOGY)

WE BUILD (CERTAIN) SPACES OUT of "CELLS" WORKING <u>UP</u> IN DIMENSION.

<u>NOTATION</u>:

D^n_α IS A COPY of $\mathbb{B}^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

S^{n-1}_α IS ITS BOUNDARY $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$.

$e^n_\alpha = D^n_\alpha - S^{n-1}_\alpha$

<u>WE BUILD X</u>

(0) $X^{(0)}$ IS A COLLECTION of POINTS EQUIPPED WITH THE DISCRETE TOPOLOGY.

(n) SUPPOSE $X^{(n-1)}$ IS GIVEN. SUPPOSE $\{D^n_\alpha\}$ IS A COLLECTION of n-CELLS. SUPPOSE $\varphi_\alpha : S^{n-1}_\alpha \longrightarrow X^{(n-1)}$ IS CONTINUOUS: φ_α IS AN <u>ATTACHING</u> MAP.

THEN THE n-SKELETON $X^{(n)}$ IS.

$$X^{(n)} = \frac{X^{(n-1)} \sqcup \bigsqcup_{\alpha} D^n_\alpha}{x \sim \varphi_\alpha(x) \left(\begin{matrix}\text{FOR } x\in S^{n-1}_\alpha \\ \text{FOR ALL } \alpha\end{matrix}\right)}$$ QUOTIENT TOPOLOGY

(∞) DEFINE $X = \bigcup_{n=0}^{\infty} X^{(n)}$, THE ASCENDING UNION.

WE EQUIP X WITH THE <u>WEAK TOPOLOGY</u>. $A \subset X$ IS OPEN IF AND ONLY IF $A \cap X^{(n)}$ IS OPEN IN $X^{(n)}$ FOR ALL n

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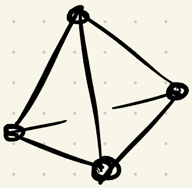
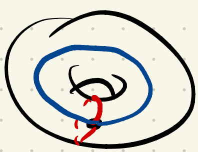
<u>EXAMPLES</u> (0) $S^0 = \{\pm 1\} \subset \mathbb{R}$. SO $X = X^{(0)}$

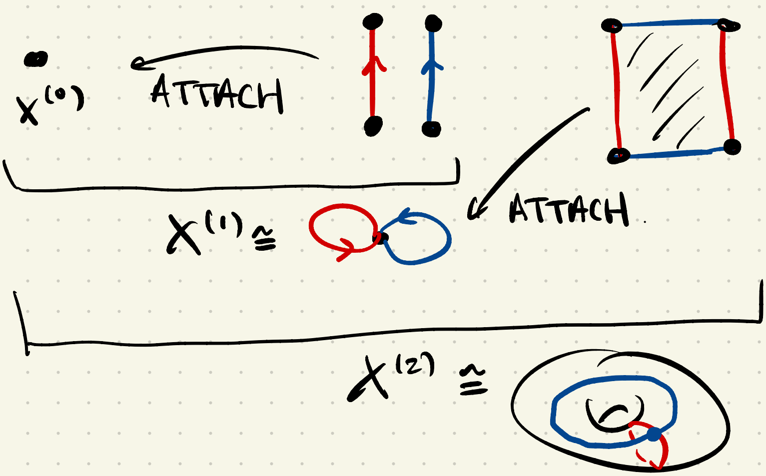
(1) 

FOUR VERTICES (ZERO CELLS)
SIX EDGES (ONE CELLS)
SO $X = X^{(1)}$.

(2) 

ONE VERTEX
TWO EDGES
ONE FACE (TWO-CELL)

SO $X = X^{(2)}$



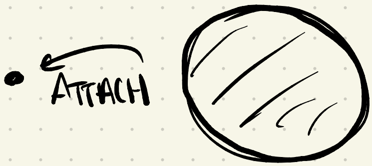
③ 

S^2

ONE ZERO CELL
ONE TWO CELL

SO $X^{(1)} = X^{(0)}$
$X = X^{(2)}$

THE ATTACHING MAP
$\varphi : S^1 = \partial D^2 \longrightarrow X^{(0)}$
IS CONSTANT



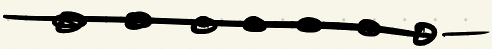
PULLING THE CORD ON A DRAWSTRING BAG.

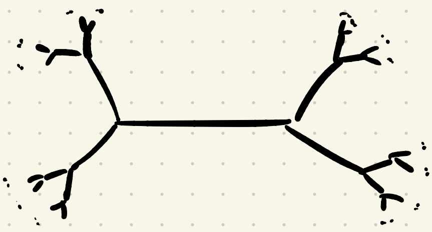
DEFINITIONS:

① X IS FINITE DIMENSIONAL IF $X = X^{(n)}$ FOR SOME n.

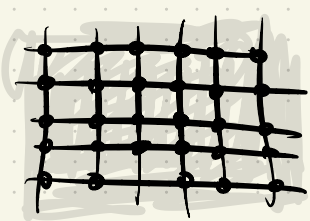
② X IS FINITE IF X HAS ONLY FIN. MANY CELLS.

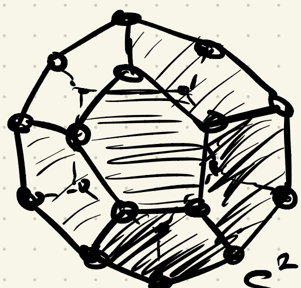
③ X IS A GRAPH IF $X = X^{(1)}$.

EXAMPLES ①  $\cong \mathbb{R}$

② 

T_3 THE REGULAR THREE VALENT TREE.

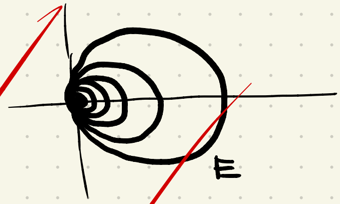
③  $\cong \mathbb{R}^2$

④ 

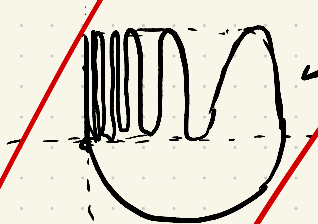
S^2

NON-EXAMPLES:

① THE EARRING SPACE


E

② THE TOPOLOGIST'S CIRCLE



LOOKS LIKE $\sin(1/x)$

③ $C = \{ (1-t)(\frac{1}{n}, 0) + t(0,1) \mid n \in \mathbb{Z},\ n > 0,\ t \in [0,1] \}$

$\cup \{ (0,t) \mid t \in [0,1] \}$