2024-11-21 SAUL SCHLEIMER LECTURE 24 MA3F1

① CW COMPLEXES ARE GREAT

EXERCISES [HATCHER, PAGE 529]

(1) SUPPOSE X IS CW-COMPLEX. SUPPOSE $p : \tilde{X} \to X$ IS COVERING MAP. THEN THERE IS A CW-COMPLEX STR ON $\tilde{X}$ SO THAT $p^{-1}(e^n_\alpha) = \bigsqcup_\beta e^n_{\alpha,\beta}$

THAT IS "CW-COMPLEX STRUCTURES LIFT TO COVERS".

(3) SUPPOSE X IS CW. THE X IS PATH CONN IF AND ONLY IF $X^{(1)}$ IS PATH-CONNECTED.

THEOREM [PAGE 97, SORT OF]. SUPPOSE X IS CW. SUPPOSE $x_0 \in X^{(0)}$. THEN $\iota : (X^{(2)}, x_0) \longrightarrow (X, x_0)$ INDUCES AN ISOMORPHISM $\iota_* : \pi_1(X^{(2)}, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$.

WE'LL RETURN TO THE PROOF.

② GENERATORS AND RELATIONS.

DEF: SUPPOSE S IS A SET. SUPPOSE $F_S = \mathop{*}_{s \in S} \mathbb{Z}$ IS THE FREE GROUP "GENERATED BY S".

TYPICAL ELEMENT: $stsst^{-1}s^{-1}s^{-1}$

SHORTHAND: $sts^2t^{-1}s^{-2}$.

SUPPOSE $R \subset F_S$. RECALL $\langle\langle R \rangle\rangle$ IS THE NORMAL CLOSURE of R: $\langle\langle R \rangle\rangle = \bigcap_{R \subset N \triangleleft F_S} N$.

NOTATION: $\langle S \mid R \rangle = F_S / \langle\langle R \rangle\rangle$ THIS IS THE GROUP GENERATED BY S AND WITH RELATIONS R

IF $G \cong \langle S \mid R \rangle$ SAY THAT $\langle S \mid R \rangle$ IS A

PRESENTATION of G. IF S IS FINITE SAY G IS FINITELY GENERATED. IF S AND R ARE FINITE SAY G IS FINITELY PRESENTED.

EXAMPLES

① $\langle 1 \rangle \cong \mathbb{1}$.

② $\langle a \mid \rangle \cong \mathbb{Z}$

③ $\langle a \mid a^n \rangle \cong \mathbb{Z}/n\mathbb{Z}$

④ $\langle a, b \mid \rangle \cong F_2$

⑤ $\langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2$

ALL FIN. PRES.

EXERCISE: ALL GROUPS HAVE PRESENTATIONS.

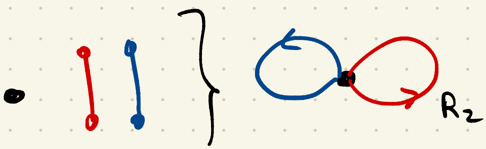
GIVE A PRESENTATION of $(\mathbb{Q}, +)$

PROVE $(\mathbb{Q}, +)$ IS NOT FIN. GEN.

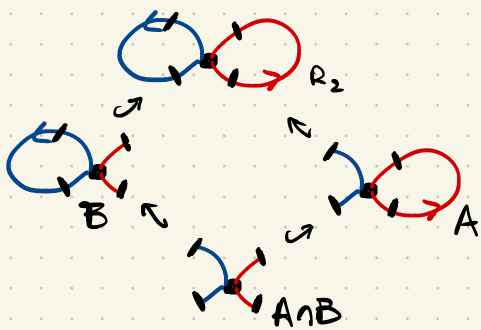
---

③ USING SvK:

THE ROSE: RECALL THAT R_2 IS THE CW-COMPLEX



WE COVER X WITH TWO OPEN SETS



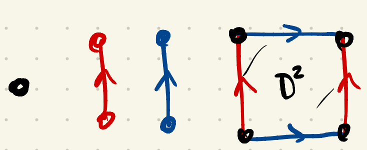
NOTE $\pi_1(A \cap B) \cong \mathbb{1}$

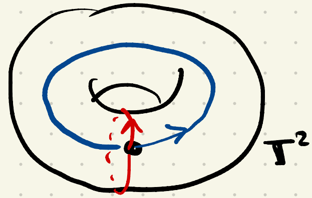
$\pi_1(A) \cong \mathbb{Z}$

$\pi_1(B) \cong \mathbb{Z}$

SO $\pi_1(R_2) \cong \pi_1(A) * \pi_1(B) / N$ WITH $N = \langle\langle \mathcal{U} \rangle\rangle$. BUT $\mathcal{U} = \varnothing$ BECAUSE $\pi_1(A \cap B) \cong \mathbb{1}$. SO $\pi_1(R_2) \cong F_2 \cong \langle a, b \mid \rangle$.

THE TORUS: RECALL $\pi_1(\mathbb{T}^2) = \pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$.





TAKING $X = \mathbb{T}^2$

$X^{(0)} = \{pt\}$

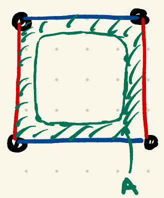
$X^{(1)} \cong R_2$

$X^{(2)} = X \cong \mathbb{T}^2$.

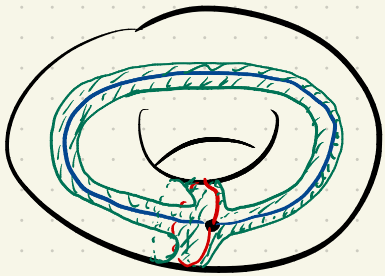
LET A BE A SMALL NEIGH. of $X^{(1)}$ IN X.

LET $B = e^2$ THE OPEN TWO-CELL. CLAIM: $A \cap B \cong S^1 \times (0,1)$.

PROOF BY PICTURES



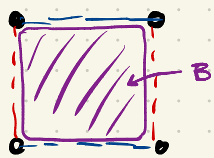
GIVES



SO $A \underset{h.e.}{\simeq} X^{(1)} \cong R_2$

SO $\pi_1(A) \cong F_2$

$\cong \langle a, b \mid \rangle$.



SO $B = X - X^{(1)} \cong$ OPEN TWO-DISK $\cong \mathbb{R}^2$

SO $\pi_1(B) \cong \mathbb{1}$.

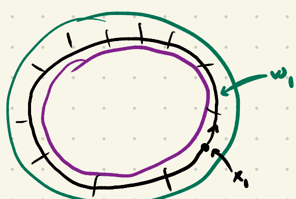
FINALLY $A \cap B \cong S^1 \times (0,1) \underset{h.e}{\simeq} S^1$

SO $\pi_1(A \cap B) \cong \mathbb{Z}$.

RECALL $\mathcal{U} = \left\{ (\iota_{AB})_*(\omega) \cdot \cancelto{e}{(i_{BA})_*(\omega^{-1})} \;\middle|\; \omega \in \pi_1(A \cap B) \right\}$

$= \left\{ (i_{AB})_*(\omega) \;\middle|\; \omega \in \pi_1(A \cap B) \right\}$ BECAUSE $\pi_1(B) \cong \mathbb{1}$.

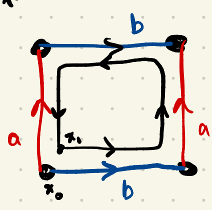
RECALL $\pi_1(A \cap B) \cong \pi_1(S^1) = \{ [\omega_n] \mid n \in \mathbb{Z} \}$

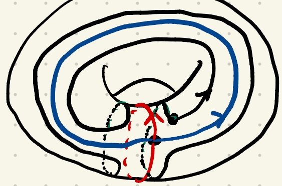


$A \cap B$

CLAIM: $(\iota_{AB})_*[\omega_1] = b a \cdot b^{-1} \cdot a^{-1}$

PICTURE:





EXERCISE: $\iota_A \circ \iota_{AB} \circ \omega_1 = \iota_B \circ \iota_{BA} \circ \omega_1 \underset{\partial}{\simeq} e$

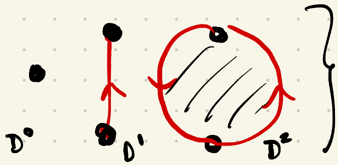
[CAREFUL: BASEPT LIES IN $A \cap B$ SO IS NOT x_0]

CLAIM: $(i_{AB})_* [\omega_n] = (b \cdot a \cdot b^{-1} \cdot a^{-1})^n$

EXAMPLE $(i_{AB})_* [\omega_2] = b \cdot a \cdot b^{-1} \cdot a^{-1} \cdot b \cdot a \cdot b^{-1} \cdot a^{-1}$.

SO: $\boxed{\pi_1(\mathbb{T}^2) \cong \dfrac{\langle a \rangle * \langle b \rangle}{\langle\langle b a b a^{-1} \rangle\rangle} \cong \langle a, b \mid b a b^{-1} a^{-1} \rangle \cong \mathbb{Z}^2.}$

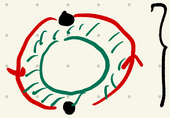
THE PROJECTIVE PLANE $\mathbb{RP}^2 = S^2 / x \sim -x$. HAS CW STR



DOES NOT EMBED IN $\mathbb{R}^3$ SO CANNOT DRAW A PICTURE!

BUT CAN COMPUTE π_1: $A =$ NEIGH of $X^{(1)}$

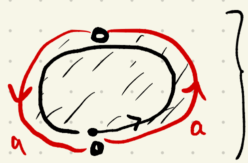
$B = e^2 \cong \mathbb{R}^2$.



MOBIUS STRIP $\underset{h.e}{\simeq} S^1$

SO $A \cap B \cong S^1 \times (0,1)$ AND $\pi_1(A \cap B) \cong \mathbb{Z}$ IS GEN BY $[\omega_1]$.

SO ω_1 GIVES



SO $(\iota_{AB})_* [\omega_1] = a \cdot a = a^2$.

THUS

$\pi_1(\mathbb{RP}^2) \cong \langle a \mid a^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

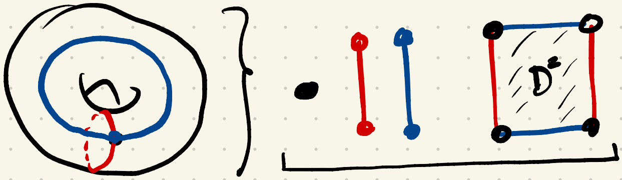
PROOF 2: $q: S^2 \longrightarrow \mathbb{RP}^2$ IS A COVERING MAP. THUS

$\pi_1(S^2) \cong \mathbb{1}$ IS INDEX TWO IN $\pi_1(\mathbb{RP}^2)$.

SO $|\pi_1(\mathbb{RP}^2)| = 2$ AND $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$.

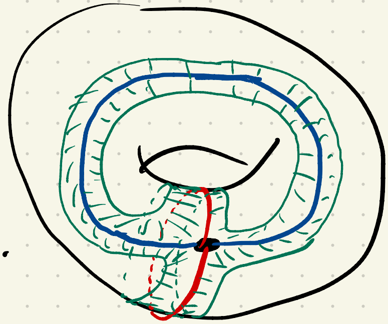
OLD VERSION of DISCUSSION of TORUS [DID NOT USE]

THE TORUS $\mathbb{T}^2$



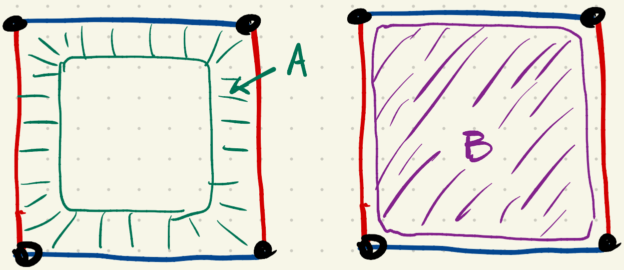
SO TAKING $X = \mathbb{T}^2$

$X^{(0)} \cong \{pt\}$, $X^{(1)} \cong R_2$. LET A BE A SMALL NEIGHBOURHOOD of $X^{(1)}$ IN X.



SET $B = D^2 - \partial D^2 = e$ THE OPEN TWO-CELL. SO $A \cap B$ IS AN OPEN ANNULUS $\cong S^1 \times (0,1)$.

PICTURE:



$$\pi_1(A) \cong \pi_1(R_2) \cong F_2 \cong \mathbb{Z} * \mathbb{Z}$$

$$\pi_1(B) \cong \pi_1(D^2) \cong 1$$

$$\pi_1(A \cap B) \cong \pi_1(S^1 \times (0,1)) \cong \pi_1(S^1) \cong \mathbb{Z}$$

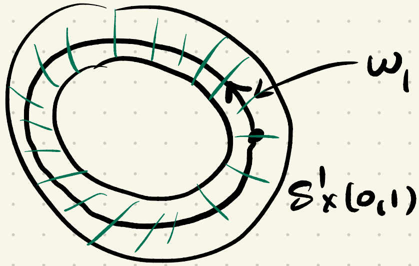
SO $\pi_1(\mathbb{T}^2) \cong \dfrac{F_2 * 1}{N}$, $N = \langle\langle \mathcal{U} \rangle\rangle$.

WITH $\mathcal{U} = \left\{ (i_{AB})_*(\omega)\,(i_{BA})_*(\omega)^{-1} \right\}$.

BUT $(i_{BA})_*$ IS TRIVIAL BECAUSE $\pi_1(B) \cong 1$.

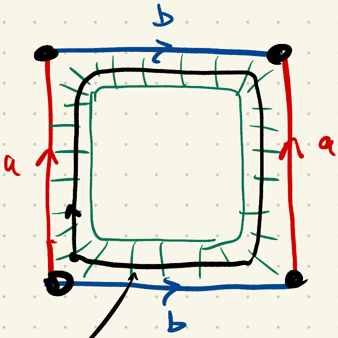
SO $\mathcal{U} = \left\{ (i_{AB})_*(\omega) \mid \omega \in \pi_1(A \cap B) \right\}$.

NOTE $[\omega_1] \in \pi_1(A \cap B)$



GENERATES $\pi_1(A \cap B)$.

THAT IS:



$$(i_{AB})_*(\omega) = a\, b\, a^{-1} b^{-1}.$$

SO:

$$\boxed{\pi_1(T^2) \cong \frac{\langle a \rangle * \langle b \rangle}{\langle\langle a \cdot b \cdot a^{-1} \cdot b^{-1} \rangle\rangle}}$$