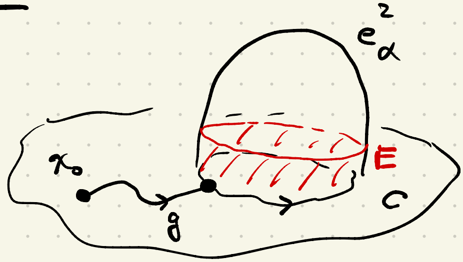
SAUL SCHLEIMER 2024-11-26 LECTURE 26 MA 3F1

① PROOF of (1.26 a)

RECALL X CW, $C \subset X$, $X - C = e^2_\alpha$ IS A 2-CELL. $\iota_C : C \hookrightarrow X$.

PICTURE

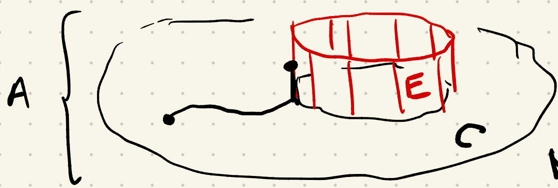


NOTE e^2_α IS A COPY of $\{ x \in \mathbb{R}^2 \mid |x| < 1 \}$.

SET
$E = \{ x \in \mathbb{R}^2 \mid \frac{3}{4} < |x| < 1 \}$
$A = C \sqcup E / \varphi_\alpha$
$B = e^2_\alpha$, SO $A \cap B = E$.

NOTE $E \cong S^1 \times (0,1)$. PICK $x_1 \in E$. NOTE $\pi_1(A, x_1) \cong \pi_1(C, x_0)$ BECAUSE A DEF. RETRACTS TO C. [AND CHANGE of BASEPT].

PICTURE



NOTE $\pi_1(B) = \pi_1(e^2_\alpha) \cong \mathbb{1}$.
AND $\pi_1(A \cap B, x_1) \cong \mathbb{Z}$.
LET $w_1 \in \pi_1(A \cap B, x_1)$ BE THE GENERATOR.
NOTE w_1 IS THE IMAGE of $\gamma = [g * \varphi|S^1_\alpha * \bar{g}]$ UNDER CHANGE of BASEPOINT.

FINALLY: APPLY SvK: $\pi_1(X, x_1) \cong \underbrace{\pi_1(A, x_1) * \cancel{\pi_1(B, x_1)}^{\,1}}_{N}$.

BUT $\pi_1(B, x_1) = \mathbb{1}$ AND $N = \langle\langle (\iota_{AB})_*(w_1) \rangle\rangle$.

WE HAVE:

$$\begin{tikzcd} C \arrow[rr, "\iota_C"] \arrow[dr, "j_C"', "\simeq\ \text{h.e}"] & & X \\ & A \arrow[ur, "\iota_A"'] & \end{tikzcd}$$

SO $\iota_C = \iota_A \circ j_C$ SO $(\iota_C)_* = (\iota_A)_* \circ (j_C)_*$

BUT $(j_C)_*$ IS ISOMORPHISM. [DETAILS ABOUT CHANGE of BASEPT]. SINCE $(\iota_A)_*$ SURJECTS WITH KERNEL N
$(\iota_C)_*$ SURJECTS WITH KERNEL $\langle\langle \gamma \rangle\rangle$. ⓐ

PROOF of (b): HERE $n \geq 3$, SO THE "SHELL" $E = \{ x \in \mathbb{R}^n \mid \frac{3}{4} < |x| < 1 \}$ HAS $\pi_1(E) \cong \mathbb{1}$.

SO $\pi_1(A \cap B) \cong \mathbb{1}$ SO H IS TRIVIAL SUBGROUP.

SO $(i_c)_*$ ALSO INJECTIVE, SO ISOMORPHISM. (b)

(c) LET $t: X^{(2)} \longrightarrow X$ BE THE INCLUSION. WTS t_* IS ISOMORPHISM ON π_1.

CASE 1: X IS FINITE: [HAS FINITELY MANY CELLS] THEN APPLY (b) AND INDUCTION.

CASE 2: X IS INFINITE.

t_* SURJECTIVE: FIX $[\gamma] \in \pi_1(X, x_0)$. SO $\gamma: I \to X$ AND $\gamma(0) = \gamma(1) = x_0$. SINCE I COMPACT, $\gamma(I)$ IS COMPACT. BY PROP (A) THERE IS SOME FINITE SUBCOMPLEX $C \subset X$ WITH $\gamma(I) \subset C$. WE HAVE A DIAGRAM of SPACES:

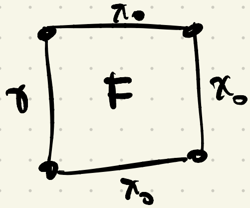
$$\begin{tikzcd} C^{(2)} \arrow[r, "s", "\cong"'] \arrow[d, "p"'] & C \arrow[d, "q"] & \\ X^{(2)} \arrow[r, "t"'] & X & I \arrow[l, "\gamma"'] \end{tikzcd}$$

ALL MAPS ARE INCLUSIONS. NOTE $x_0 \in C^{(2)}, C, X^{(2)}, X$. BY (b), $s_*: \pi_1(C^{(2)}, x_0) \to \pi_1(C, x_0)$ IS AN ISOM.

BY CONSTRUCTION: $q_*[\gamma] = [\gamma]$. LET $[\gamma'] = s_*^{-1}([\gamma])$.
SO $\gamma' \in \text{LOOPS}(C^{(2)}, x_0)$ AND $\gamma' \simeq \gamma$ VIA HOMOTOPY IN C.
FINALY, SINCE $q_* s_* [\gamma'] = [\gamma]$ WE HAVE

$t_* p_* [\gamma'] = [\gamma]$ SO t_* SURJECTS.

t_* INJECTIVE: SUPPOSE $[\gamma] \in \pi_1(X^{(2)}, x_0)$ AND $t_*[\gamma] = [e] \in \pi_1(X, x_0)$. SO $\gamma \simeq e$ IN X VIA HOMOTOPY $F : I \times I \longrightarrow X$.



NOTE $I \times I$ COMPACT, SO $F(I \times I)$ IS COMPACT. SO, BY PROP A1 THERE IS A FINITE SUBCOMPLEX $C \subset X$ WITH $F(I \times I) \subset C$. [SO $\gamma(I) \subset C$ AS WELL!]. AGAIN

$$\begin{tikzcd} C^{(2)} \arrow[r, "s", "\cong"'] \arrow[d, "p"'] & C \arrow[d, "q"] \\ X^{(2)} \arrow[r, "t"] & X \\ & I \times I \arrow[u, "F"'] \end{tikzcd}$$

HAVE DIAGRAM of SPACES.
SO $[\gamma] = [e]$ IN $\pi_1(C)$.
AGAIN TAKE $[\gamma'] = (s_*)^{-1}[\gamma]$.
SO $[\gamma'] = [e]$ IN $\pi_1(C^{(2)})$.
THUS $p_*[\gamma'] = [\gamma]$ IS TRIVIAL IN $\pi_1(X^{(2)})$, AS DESIRED. ☺

PROP 1.26

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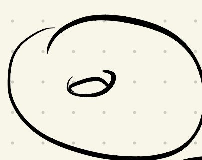
(2) THE SURFACE of GENUS TWO

DEF: A SURFACE IS A SECOND COUNTABLE, HAUSDORFF SPACE F^2 WHERE EVERY $x \in F^2$ HAS A NEIGH $U \subset F^2$ WHICH IS HOMEO. TO $\mathbb{R}^2$.

EXAMPLES: $\mathbb{R}^2, S^2, T^2, \mathbb{RP}^2$.

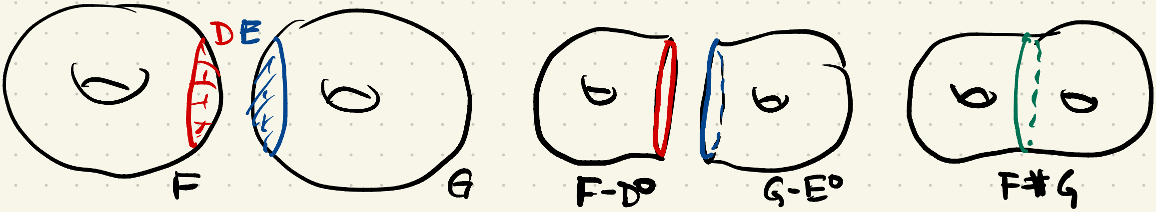
S^2



T^2.

WE CAN MAKE NEW SURFACES FROM OLD BY **CONNECT SUM**: SUPPOSE F, G SURFACES. PICK DISKS $D \subset F$, $E \subset G$. SET $D^\circ = D - \partial D$, $E^\circ = E - \partial E$. FIX HOMEO $\varphi : \partial D \to \partial E$. DEFINE $F \# G = \dfrac{(F - D^\circ) \sqcup (G - E^\circ)}{x \sim \varphi(x)}$

PICTURE



$F \# G$ IS THE **CONNECT SUM** of F AND G.

GOAL: GIVE A PRESENTATION of $\pi_1(F \# G)$.

EXERCISE: $T^2 \# \mathbb{RP}^2 \cong \#_3 \mathbb{RP}^2$.

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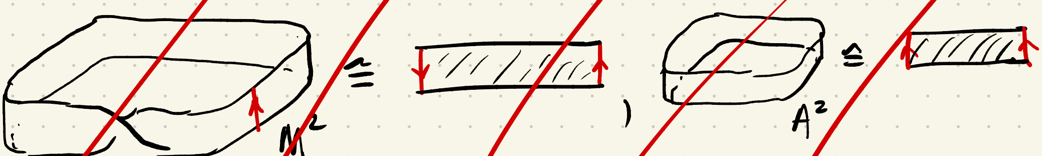
③ **SURFACES WITH BOUNDARY** WE ALLOW POINTS TO HAVE NEIGHBORHOODS HOMEO TO $\mathbb{R}^2_{\geq 0} = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$

EXAMPLES: $\mathbb{R}^2_{\geq 0}$, D^2, $A^2 = S^1 \times [0,1]$, M^2



EXERCISE: $\mathbb{RP}^2 \cong M^2 \cup_\partial D^2$ [GLUE ALONG BOUNDARY].